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Nash equilibrium when players account for the complexity of their forecasts

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Abstract

Nash equilibrium is often interpreted as a steady state in which each player holds the correct expectations about the other players' behavior and acts rationally. This paper investigates the robustness of this interpretation when there are small costs associated with complicated forecasts. The model consists of a two-person strategic game in which each player chooses a finite machine to implement a strategy in an infinitely repeated 2×2 game with discounting. I analyze the model using a solution concept called Nash Equilibrium with Stable Forecasts (ESF). My main results concern the structure of equilibrium machine pairs. They provide necessary and sufficient conditions on the form of equilibrium strategies and plays. In contrast to the "folk theorem," these structural properties place severe restrictions on the set of equilibrium paths and payoffs. For example, only sequences of the one-shot Nash equilibrium can be generated by any ESF of the repeated game of chicken.

JEL classification: 021; 022; 026

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1. Introduction

Introducing the notion of Nash equilibrium in their text book, Osborne and Rubinstein (1994, p. 14) write: "*The most commonly used solution concept in game theory is that of Nash equilibrium. This notion captures a steady state of the play of a strategic game in which each player holds the correct expectation about the other players' behavior and acts rationally*."

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The above citation describes one of the most commonly accepted interpretations of Nash equilibrium. It says that the equilibrium strategy of a player represents not only the action plan this player actually takes, but also this player's plan of action as envisioned by the other players. According to this interpretation, players' strategies in a Nash equilibrium must meet *two* requirements:

(1) they must be best responses to each other, and
(2) they must also represent what the other players expect each player to do.

Thus, if for some reason player i's actual strategy does not coincide with player j's expectations, then we are not at a Nash equilibrium.

This paper investigates the robustness of the above interpretation of Nash equilibrium, when there are small costs associated with complex forecasts. The paper addresses the following question: What is the set of strategy profiles that retain the two properties described above, when players try to use the simplest forecasts? I argue that this set can be surprisingly small.

In order to address the question posed above I perform the following exercise for two-person games. I look at the Nash equilibria of a game and then ask, if complicated forecasts are costly, will each player continue to maintain an accurate forecast of his opponent? Suppose one of the players can find a best response to his opponent, which is rationalized by a simpler (but possibly inaccurate) forecast. Then the original pair of strategies cannot be considered a Nash equilibrium that is consistent with our interpretation of this solution concept.

To help motivate the question this paper addresses, consider the following example. An army is engaged in (the strategic form of) the infinitely repeated game of chicken. The stage game payoffs are given in Fig. 1.

You are an intelligence officer in charge of analyzing the opponent and reporting your forecast of his strategy to the Chief of Staff (COS). Given your forecast, the COS will choose a best response. The COS's are replaced every period, and every new COS requests an intelligence report on the opponent. The intelligence report you prepare must pass through a long chain of hierarchy before arriving at the COS's desk. That is, you pass your report to the officer in charge of you, who edits it and then sends it to the officer in charge of him who edits your officer's report and so forth. Along this chain of command there are many opportunities for your report to be distorted so that the final version of it (which the COS receives) may be very different from your original report.

Suppose you come to the conclusion that the opponent is using a "grim trigger-strategy": He starts by cooperating and continues to do so as long as you cooperate as well; if you defect at any period, he will forever defect. You conclude that the best response is to cooperate in every period. However, you worry that a report describing the threat might get distorted along the chain of command (for example, someone may decide to

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(0, 0)$

Fig. 1. The game of chicken.

simplify it by writing "The opponent always cooperates"). Therefore, you consider sending a simple report, which has a negligible chance of getting misunderstood: "The opponent always defects." A very simple best response to this report is to always cooperate. This is also a best response to what you believe to be the opponent's actual strategy. Thus, you decide to report that the opponent always defects, expecting this report to reach the COS without mistakes. Since the forecast you send mentions nothing about a possible defection of his opponent, it seems natural that the COS will not decide to threaten his opponent with a punishment. Thus, you believe the COS will respond to the report by constantly cooperating, thus obtaining the highest payoff against his opponent.[1]

The above argument suggests that a pair of grim trigger-strategies may not qualify as a Nash equilibrium when the complexity of forecasts is taken into account. Moreover, since trigger-strategies seem intuitively very simple, the example suggests that more complicated strategies, which enforce constant cooperation, may also fail to qualify as Nash equilibria.

The example above introduces a central theme of this paper, which is that expectations that can be simplified without affecting the players' payoffs, will eventually be replaced. This assumption is motivated by the observation that complicated descriptions of strategies have undesirable features. For example, complicated descriptions tend to be more difficult to understand and more difficult to remember. Furthermore, complicated descriptions may be more costly to communicate: They take longer to explain and stand a higher chance of being distorted along the communication channel (especially when the description must pass through several hands until it reaches its final destination).

One may study the interaction of agents who account for the complexity of their forecasts in the context of any model. It is particularly appropriate in the context of an extensive game, where a forecast determines the opponent's actions in various circumstances. This allows the use of forecasts that are intuitively very simple, as well as those that are intuitively very complicated. Within the set of extensive games, I study the model of infinitely repeated 2×2 symmetric games with discounting. This model includes many well-known games which have been widely studied, such as the Prisoner's Dilemma, Chicken, and the Battle of the Sexes. Following the literature on complexity considerations in repeated games (in particular, Rubinstein (1986), Abreu and Rubinstein (1988), and Piccione (1992)), I analyze the strategic game in which players choose finite machines to implement their strategies in the repeated game.

The use of machines allows one to model simplicity in ways which are relatively intuitive. Simplicity is defined in this model by a partial ordering over machines. Loosely speaking, one machine is said to be simpler than another if the behavior of the first machine is less dependent on the actions of the opponent, or if the first machine has less changes in its modes of behavior, or both.

I analyze the model using a solution concept called Nash Equilibrium with Stable Forecasts (*ESF*). This solution concept says that a pair of machines is an *ESF* if two conditions are satisfied:

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[1] A natural question that may arise, is why cannot cooperation be sustained if each player reports that his opponent always defects? If the COS of each army employs an intelligence officer who reasons as you have, then each COS would cooperate in each period independently of his opponent's actions. But then each intelligence officer would want to report the truth, i.e., that the opponent is cooperating each period.

(1) the machine of each player is a best response to his opponent's machine, and
(2) no player has a best response to his opponent, which can be rationalized by a forecast, which is simpler than his opponent's machine.

This definition is meant to capture our interpretation of Nash equilibrium in a world where players may not want to use correct forecasts which are too complicated. The definition of *ESF* is also motivated by the interpretation of equilibrium strategies in extensive games as representing not only the player's plan of action, but also his opponents' beliefs about him (see Aumann (1987) and Rubinstein (1991)).

My main results concern the structure of equilibrium machine pairs. They provide necessary and sufficient conditions on the form of equilibrium *strategies* and *plays*. These structural properties place restrictions on the set of equilibrium paths and payoffs. For example, only sequences of the one-shot Nash equilibrium (outcomes on the off-diagonal) can be generated by any *ESF* of the repeated game of Chicken. I also show that there are only two possible *ESF* play paths in the repeated Prisoner's Dilemma: Either both players defect each period, or both cooperate each period. Given the restrictions on play paths, I characterize the set of equilibrium payoffs.

The above results should be contrasted with those for the usual repeated games for which the Nash equilibrium set is very large in the space of strategies as well as in the space of outcomes. In particular, the "folk theorem" (see Fudenberg and Maskin (1986)) applies: All individually rational outcomes can be generated by some pair of Nash equilibrium strategies. However, when the cost of forecasts enter the players' preferences even lexicographically, the set of equilibrium outcomes is drastically reduced.

I interpret my results as a critique of our interpretation of Nash equilibrium. The "correct expectations" interpretation may suit games in which the formation or use of forecasts is costless. However, as soon as players acknowledge that complicated forecasts may be misunderstood, or as soon as we introduce a small cost for forming or maintaining complicated forecasts, then the correct expectations interpretation places severe restrictions on the set of equilibrium outcomes.

This paper is organized as follows. Section 2 discusses the related literature. The model is introduced in Section 3. Section 4 provides the equilibrium characterization, and in Section 5 I characterize the equilibrium payoffs. This is followed by concluding remarks.

2. Related literature

This paper is closely related to the literature on strategic complexity in repeated games and in particular to the works on finite automata. Among the many works in this literature, the papers most closely related to mine are Abreu and Rubinstein (1988) (henceforth denoted AR) and Piccione (1992). Central to most of the works in this literature is the assumption, which is absent in this paper, that the players' preferences are negatively affected by the complexity of their own strategy. The novel feature of this paper is the assumption that each player's *expectations* about his opponent are affected by the complexity of his opponent's strategy.

The only paper I am aware of that accounts for the complexity of describing the opponent's strategy is Spiegler (2001). Spiegler studies two-person extensive form games in which each player needs to justify (ex-post) his choice of strategy by offering a hypothesis on what the opponent's strategy is. The opponent's hypothetical strategy needs to be the simplest strategy, which is consistent with the observed history of play.

Aside from incorporating complexity considerations into the determination of a player's beliefs about his opponent, the two papers, Spiegler's and the present one, are completely different. Spiegler's goal is to propose a procedurally rational solution concept for extensive games in which players care not only about their material payoffs, but also on whether their choice of strategy can be justified ex-post to a third party. The solution concept suggested by Spiegler is very much different from an *ESF*. The reader is referred to his paper for further details.

The literature on *Psychological Games* (in particular, Dufwenberg and Kirchsteiger (2000) and Geanakoplos et al. (1989)) is a related strand of literature which model games in which the players' preferences are affected by their beliefs (in particular, the players' beliefs about the other players' beliefs). However in this literature the players do not choose their beliefs, but rather the players' beliefs are derived in equilibrium where they are assumed to be correct.

In Eliaz (2001) I explicitly model a situation in which advisors take into account the possibility that their advise will not be understood. That paper considers a game between two organizations, each consisting of a decision maker and an advisor. The decision maker receives a forecast from his advisor to which he best responds. The advisor, who observes the strategy of the opponent, has a probabilistic belief on the mistakes his decision maker can make. In equilibrium, each advisor sends a forecast that maximizes the decision maker's expected payoffs, given the strategy of the opponent decision maker and given the advisor's beliefs over the mistakes his decision maker can make.

3. A model

Let G be a 2×2 symmetric game. The set of actions available to each player is $A = \{C, D\}$. We use the notation a_i to refer to an action taken by player i, and we let $-a_i$ denote that player's other action (i.e., for each $a_i \in \{C, D\}$, $-a_i \in \{C, D\}\backslash\{a_i\}$). A G-outcome is a member of A^2 and is denoted by $\mathbf{a}$ such that $\mathbf{a} = (a_1, a_2)$. To save on notation, we suppress subscripts whenever it is clear which player is carrying out the action, or whenever the identity of the player is unimportant (such as when both players choose the same action). Thus, the G outcomes in which both players choose the same action are (D, D) and (C, C), whereas the outcome in which player i chooses the action D while player j chooses C is (D_i, C_j).

Each player's payoff is represented by a utility function $u : A^2 \to \Re$ with u_i denoting the payoff to player i. We let D be each player's minmax action, i.e., D solves $\min_{a_i} \max_{a_j} u_j(a_1, a_2) \equiv v_j$. We assume that $u(C, C) \neq u(D_i, C_j)$ and $u(C_i, D_j) \neq u(D, D)$. The class of games which satisfy these conditions include many well-known examples such as the Prisoner's Dilemma, Chicken, and the Battle of the Sexes. Figure 2 displays examples of payoff matrices for those games.

	C	D
C	(3, 3)	(0, 4)
D	(4, 0)	(1, 1)

(a)

	C	D
C	(3, 3)	(1, 4)
D	(4, 1)	(0, 0)

(b)

	C	D
C	(0, 0)	(1, 3)
D	(3, 1)	(0, 0)

(c)

Fig. 2. (a) The prisoner's dilemma. (b) Chicken. (c) The battle of the sexes.

Each player i evaluates a sequence of G-outcomes $(\mathbf{a}^t)$ by applying the discounting criterion to the induced sequence of utility numbers $(u_i(\mathbf{a}^t))$. We refer to $(1-\delta)\times\sum_t \delta^{t-1}u_i(\mathbf{a}^t)$ as player i's repeated game payoff in the repeated game with discounting.

We study a static version of a repeated game in which each player i chooses a finite machine $m_i \in M_i$ to play the infinitely repeated game of G (see Osborne and Rubinstein (1994, pp. 140–143)) for an introduction on the use of machines as a modeling device in repeated games). We refer to this strategic game as a "machine game" denoted by $MG(\delta)$. A machine of player i is a four-tuple $(Q_i, q_i^0, f_i, \tau_i)$ where Q_i is a finite set of states, q_i^0 is the initial state, $f_i : Q_i \to A$ is an output function that assigns an action to every state and $\tau_i : Q_i \times A \to Q_i$ is the transition function that assigns a state to every pair of a state and an action of the other player. We say that the transition from a state is *constant*, if it is independent of the other player's actions, that is, if $\tau_i(q_i, C) = \tau_i(q_i, D)$. We refer to the triple (Q_i, q_i^0, τ_i) as the automaton's *architecture*. The preferences of the players over possible pairs of machines are as follows. Player i prefers the pair of machines (m_1, m_2) to the pair (m_1', m_2') if and only if he prefers the induced sequence of outcomes $(\mathbf{a}^t(m_1, m_2))_{t=1}^{\infty}$ to the sequence $(\mathbf{a}^t(m_1', m_2'))_{t=1}^{\infty}$.

In the sections that follow we will often refer to states in the players' machines that "appear at certain periods" on the play path that is generated by those machines. For this we shall use the following notations. Given a pair of machines m_1 and m_2 we let q_i^t denote the state that m_i is at in the tth period of $\mathbf{a}(m_1, m_2)$. The set of states that m_i "passes through" from period t to $t+k$ will be denoted by $Q_i(t, t+k)$; that is, $Q_i(t, t+k) = \{q_i^t, \ldots, q_i^{t+k}\}$.

3.1. Simplicity

We now define a partial ordering of machines which we interpret as the ranking of a player's forecasts according to their level of complexity. This ordering is assumed to depend only on the machines' architecture. Let m be a machine with a set of states Q and a transition function τ. We denote by $x(m)$ the number of distinct pairs of states in Q that are connected by a transition. That is, $x(m)$ equals the number of pairs (q, q') in Q such that $q \neq q'$ and $q' = \tau(q, a)$ for some $a \in \{C, D\}$. For example, the machine m depicted in Fig. 3 below satisfies $x(m) = 1$.

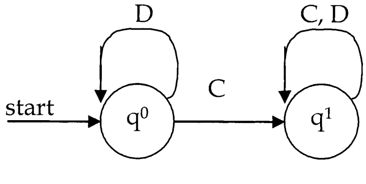

Fig. 3. Machine m.

We shall call $x(m)$ the "number of transitions from one state to a different state."

Similarly, we let $y(m)$ denote the number of states in Q with the property that when m reaches each of those states, the next state it visits depends on the other player's action:

$$y(m) = \left|\left\{q \in Q\colon\ \tau(q, C) \neq \tau(q, D)\right\}\right|.$$

Thus, $y(m)$ measures the number of non-constant transitions in m. For example, in Fig. 3, $y(m) = 1$.

Definition 1. Let m and m' be a pair of machines. We say that m is *simpler* than m' whenever $(x(m), y(m)) < (x(m'), y(m'))$.

The above measure of complexity is meant to capture the intuition that complex descriptions are either more difficult to remember, or a have a higher probability of being distorted (in particular, a complicated description that needs to pass many hands until it reaches its final destination has a higher probability of getting distorted along the way). The intuition for the assumption that the simplicity of a (forecast of the) machine m is inversely related to $y(m)$ is the following: The less conditionals a strategy contains, the easier it is to remember and the less likely it is to be distorted. For example, the description "Player j chooses b only if you choose a" might be confused with "Player j chooses b unless you choose a" or with "Player j chooses a only if you choose b." However the statement "Player j chooses b" is easier to remember and less likely to be distorted.

If $x(m) < x(m')$, then a forecast that describes the opponent as using m is simpler than a forecast in which the opponent's machine is m' in the sense that the description of the opponent's strategy is shorter in the former forecast than in the latter. For example, suppose you are describing some state of behavior of the opponent. If one of your actions, say a, would cause the opponent to move to a different state of behavior, then you would also need to describe that state of behavior in addition to the one you are describing now. A much shorter description is to say that when you choose a your opponent would remain in the state which you have just described.

3.2. An equilibrium notion

We now define a Nash equilibrium of $MG(\delta)$ which is immune to forecast simplification. A Nash equilibrium having this property satisfies the following: Any player who best responds to a simpler machine than his opponent's, necessarily reduces his payoffs in the game.

Definition 2. A Nash equilibrium with stable forecasts (*ESF*) of a machine game $MG(\delta)$ is a pair of machines (m_1^*, m_2^*) with the following properties: For every player i,

(1) m_i^* is a best response to m_j^*, and
(2) for any machine m_j simpler than m_j^*, a best respond to m_j is not a best respond to m_j^*.

The definition of *ESF* is motivated by the interpretation of Nash equilibrium as *a steady state where each player best responds to an accurate forecast of his opponent*. According to this interpretation, the Nash equilibrium strategy of player i not only represents this player's best response to j, but also represents player j's belief about player i's strategy.

The solution concept we offer is one answer to the following question. Suppose there is a tendency to simplify forecasts, whenever the simplification does not reduce payoffs. Which strategy pairs will remain a Nash equilibrium? That is, which pairs of strategies retain the property that *each player best responds to a correct forecast of his opponent's strategy*. If a player can obtain the maximal payoffs against his opponent by best responding to a forecast, which is simpler than his opponent's strategy, then the strategies of these two players are not a Nash equilibrium according to our interpretation.

An *ESF* may be interpreted as a Nash equilibrium when players have lexicographic preferences in which the simplicity of the forecast is secondary to material payoff. Such preferences imply that a player prefers to rationalize his payoff-maximizing strategy with the simplest forecast. Lexicographic preferences represent a conservative way of allowing the simplicity of forecasts to enter into players' preferences in the sense that the simplicity of a forecast never outweighs material payoffs. Lexicographic preferences for simple forecasts can be interpreted as the preferences of a player who receives his forecast from someone who observes his opponents and (a) would like the player to succeed, but (b) is concerned that complicated forecast may not be understood. Lexicographic preferences can also be interpreted as representing evolutionary forces which favor players who use the simplest forecasts to obtain the highest payoffs against their opponent.

It is instructive to note that there exists an *ESF* in any repeated game in which there exists a pure strategy Nash equilibrium in the constituent game. To see why, let G be a strategic game with a pure Nash equilibrium (a_1^*, a_2^*). Let m_i^* be a machine with a single state in which player i plays a_i^*. The pair of machines (m_1^*, m_2^*) is a Nash equilibrium of $MG(\delta)$ for all $\delta \in [0, 1]$. Since there is no simpler machine than a single-state machine, it follows that (m_1^*, m_2^*) is also an *ESF* of $MG(\delta)$.

To understand how *ESF* can be applied to repeated games we look at two simple examples. Each example considers a particular Nash equilibrium of the repeated Prisoner's Dilemma. The equilibrium in the first example is shown to satisfy the requirements for *ESF*, whereas the equilibrium in the second example violates those requirements.

Example 1. Suppose G represents the Prisoner's Dilemma depicted in Fig. 2. By the Nash folk theorem there exists a discount factor $\delta^* \in (0, 1)$ such that for all $\delta \geqslant \delta^*$ when both players use the machine m_i (see Fig. 4), we have a Nash equilibrium of $MG(\delta)$ in which the players cooperate in every period.

We claim that (m_1, m_2) is also an *ESF* of $MG(\delta)$ for all $\delta \geqslant \delta^*$. To see why note that it is optimal for a player to cooperate only if his opponent threatens him with a punishment, and

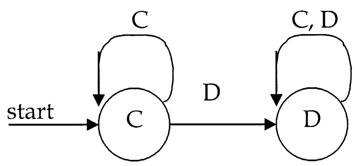

Fig. 4. The machine m_i.

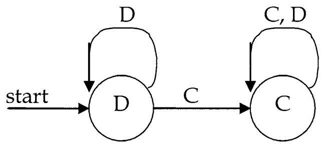

Fig. 5. The machine m'_i.

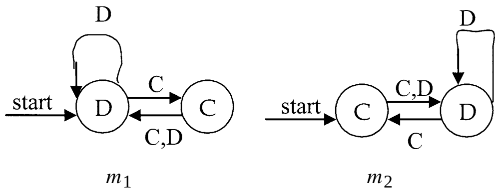

Fig. 6.

only if his opponent's machine has at least two states (one, which "rewards" cooperation and one which punishes defection). The only machine, which is simpler than m_j and has two states, is one with constant transitions. However, given such a forecast player i's best response is to choose D every period. Therefore, any forecast, which induces player i to choose C every period cannot be simpler than m_j. Note that for a discount factor sufficiently close to one, m_i is also a best response to m'_j, depicted in Fig. 5. However, m'_j is not simpler than m_j.

In the next example we consider again the infinitely repeated Prisoner's Dilemma. We show that for any discount factor, a pair of machines, which generates a cycle of $((D, C), (C, D))$ is not an *ESF*.

Example 2. Consider the pair of machines (m_1, m_2) depicted in Fig. 6.

For δ sufficiently close to one, (m_1, m_2) is a Nash equilibrium of $MG(\delta)$. However, this pair does not constitute an *ESF* for any discount factor. To see why, note that m_1 is also a best response to a forecast, in which player 2 uses a machine with a single state of defection.

3.3. Discussion

Before proceeding to the results, we discuss the motivation for studying our proposed solution concept. Our motivation stems from the following interpretations of our model.

3.3.1. A critique of Nash equilibrium

The focus of this paper is on Nash equilibrium and how it is affected when we introduce small costs for maintaining complicated forecasts. By Nash equilibrium I mean a steady state in which each player has a correct forecast of his opponent's strategy and chooses a best response to that forecast. One natural way to extend the standard definition of Nash equilibrium to a world with costly forecasts is to add the requirement that no player should have an incentive to best respond to an incorrect forecast. This is the approach we take in defining an *ESF*.

Of course, it is also interesting to check if other solution concepts, which do not require forecasts to be correct (like self-confirming equilibrium), are also sensitive to the standard assumption that forecasts are costless to form. This paper takes the view that before we investigate the many existing solution concepts, we should start with the basic notion of Nash equilibrium.

3.3.2. Decision makers who rely on the forecast of an expert

Decision makers often rely on experts to provide them with information about their competition. Most experts would like to help their clients make optimal decisions. However, it is not uncommon for experts to be concerned with the costs (time, mistakes, etc.) associated with complex forecasts. Some examples of decision makers and experts who fit the above description include the following: intelligence services, which provide information to operational units; market specialists who provide top management with information about competitors; a congressional committee, which needs to decide on a strategy against some opponent (be it terrorists, illegal aliens, drug dealers, etc.) and therefore summons an expert to inform them of the opponent's behavior.

The approach we take in this paper can be interpreted as a *reduced form* of a model that attempts to capture the strategic interaction between players who rely on expert forecasters with the above concerns. We view our proposed solution concept as the steady state of such a model. This view is based on our conviction that the following must be true in a steady state:

(1) experts have no incentive to simplify their forecasts, and
(2) experts do not provide false forecasts.

One limitation of our approach is that the full blown model in which the players and experts interact is left in the background. Thus, an important extension of our present work would be to explicitly model the decision makers and their expert forecasters. One step towards this direction is taken in Eliaz (2001).

3.3.3. Complexity considerations in repeated games

ESF is in some sense complementary to the solution concepts that have been studied in the literature on complexity considerations in games. *ESF* is the "flip side" of those solution concepts in the following sense: Instead of assuming that a player prefers simple strategies but is indifferent to forecasts of varying complexity, we assume that the player prefers simple forecasts but is indifferent to strategies of varying complexity. This allows us to isolate the effect of preferring simple forecasts from the effect of preferring simple strategies. As we show in the next section, both approaches to complexity considerations result in a one-to-one correspondence between states and actions on the equilibrium path; however the two approaches imply different sets of equilibrium play paths.

4. Characterization of the equilibria

In this section we present the main properties of *ESF* in infinitely repeated 2×2 games. We start by describing the main features of the equilibrium machines. In particular, the nature of the states on and off the equilibrium path is described. We conclude by providing the conditions that are necessary and sufficient for a sequence of action pairs to be sustainable in *ESF*.

The results in this section rely on the Markovian nature of each player's optimization problem in the machine game. Denote by $U_i(m_1, m_2)$ the repeated game payoff of player i if the players use the machines m_1 and m_2. Having $m_i^* = (Q_i, q_i^0, f_i, \tau_i)$, for each $q \in Q_i$, let $V_j(q) = \max_{m_j} U_i(m_i^*(q), m_j)$, where $m_i^*(q)$ is the machine that differs from m_i^* only in the initial state, q. For each $q \in Q_i$, let $A_j(q)$ be the set of solutions to the problem:

$$\max_{a_j \in A_j} u_j\big(f_i(q), a_j\big) + \delta V_j\big(\tau_i(q, a_j)\big).$$

Lemma 1. *Player j's machine is a best response to m_i^* if for every $q \in Q_i$, the action he takes when player i's machine is in state q is a member of $A_j(q)$. Conversely, if for every state q that m_i^* reaches, player j takes an action in $A_j(q)$, then player j's machine is a best response to m_i^*.*

Proof. See Rubinstein (1998, p. 153). □

The first result is concerned with periods on the equilibrium path in which it is optimal for a player to choose the best response in the 2×2 game to his opponent's action in those periods. In periods that have this property a player does not need to be "threatened" in order to induce him to take the right action. That is, if t is a period on the equilibrium path that has this property, then to take the optimal action at t a player can simply believe that his opponent's machine would move to its next state independently of his actions.

Proposition 1 (no unnecessary punishments). *If (m_1^*, m_2^*) is an ESF of the game MG(δ), then for every period t along the equilibrium path in which player i chooses a G-best response to his opponent's action at t, the transition of m_j^* is constant. That is,*

$$\tau_j\big(q^t, a_i\big) = \tau_j\big(q^t, -a_i\big)$$

Proof. Suppose there is a period t^* in which player i's action, a^*, is a G-best response to his opponent. Assume that $\tau_j(q^*, a_i) \neq \tau_j(q^*, -a_i)$, where q^* is the state of m_j^* at t^*. Let m_j' denote a machine, which is derived from m_j^* by letting the transition from state q^* be equal to $\tau_j(q^*, a^*)$ independently of player i's actions (any state, which cannot be reached from the initial state for any sequence of actions by player i is deleted). Thus, m_j' is simpler than m_j^*. By Lemma 1, $a^* \in A_i(q^*)$. It follows that m_i^* is a best response to m_j', a contradiction. □

Proposition 1 can be interpreted as saying that if it is not crucial for a player to know that his opponent is threatening him, then it is better to ignore the threat. Describing the opponent's threat will not change the player's incentives and may only confuse the player as the description of the opponent's strategy becomes more complex. Thus, in periods on the equilibrium path, in which a player need not know about a threat of punishment, his opponent will not threaten.

However, there are situations in which a player would not choose the correct actions, unless he knows that his opponent would punish him for not choosing correctly. The next proposition addresses these situations. It says that if a player must know that his opponent is threatening him with a punishment, then he should consider only the simplest effective punishment. This implies that off-equilibrium punishments have a very simple structure.

Proposition 2 (characterization of states not on the equilibrium path). *Let (m_1^*, m_2^*) be an ESF of MG(δ). If there is a player i and a state q_i in m_i^* such that there exists no period t that satisfies $q_i^t(m_1^*, m_2^*) = q_i$, then q_i has the following properties*:

(1) $f_i(q_i) = D_i$;
(2) $\tau_i(q_i, a_j) = \tau_i(q_i, -a_j) = q_i$.

Proof. Let t^* denote a period on the equilibrium path in which player j's machine is at state q^* and player i chooses the action a^*. Suppose there is no period on the equilibrium path in which the state of m_j^* is $\tau_j(q^*, -a^*)$. By Proposition 1, a^* is not a G-best response to $f_i(q^*)$. If Condition 2 is not satisfied, then a forecast, which satisfies the two conditions of the proposition, is simpler than m_j^*. Furthermore, by Lemma 1, m_i^* is a best response to this forecast. If only Condition 1 is not satisfied, then it is optimal for player i to choose $-a^*$ at t^*, a contradiction. □

The implication of Proposition 2 for the infinitely repeated Prisoner's Dilemma, for example, is that any off equilibrium state must be a grim-trigger threat of constant defection.

The previous propositions have characterized the threats and punishments in each player's strategy. That is, the states and transitions that are not used in equilibrium. We now turn to discuss the properties of the equilibrium path.

Because each player's machine is finite, there is a minimal number t' such that for some $t > t'$, we have $q_i^t = q_i^{t'}$ for both $i = 1$ and $i = 2$. Let t^* be the minimal such t. The sequence of pairs of states starting in period t' consists of cycles of length $t^* - t'$. We refer to this sequence as the *cyclic phase*; the sequence before period t' is the *introductory phase*.

The next proposition shows that the set of states a player uses in the cyclic and introductory phases are disjoint. This means that in an *ESF* the introductory phase of each machine consists of states that are visited only once on the equilibrium path, whereas the cyclic stage consists of those states that appear every l periods on the equilibrium path, where l is the cycle's length.

Proposition 3 (the equilibrium path consists of disjoint states). *Let (m_1^*, m_2^*) be an ESF of MG(δ) for some discount factor δ. Then for every player i there exists an integer*

$l_i < t_i^*$ such that the states in the sequence $(q_i^t(m_1^*, m_2^*))_{t=1}^{t_i^*-1}$ are disjoint and $q_i^t(m_1^*, m_2^*) = q_i^{t-l_i}(m_1^*, m_2^*)$ for $t \geqslant t_i^*$.

Proof. Let t_j be the first period on the equilibrium path in which one of the states of $m_j{}^*$ appears for the second time (by the finiteness of m_j^* such a period exists). That is, there is an integer $l_j < t_j$ that satisfies $q_j^{t_j-l_j} = q_j^{t_j} \equiv q_j^*$.

Assume that $q_j^{t_j+1} \neq q_j^{t_j-l_j+1}$. Let $(a_i^{t_j-l_j}, a_j^{t_j-l_j}) = (a_i^*, a_j^*)$. Therefore, $a_i^{t_j} = -a_i^*$ and $\tau_j(q_j^*, C) \neq \tau_j(q_j^*, D)$. Let m'_j denote the machine, which is obtained by changing the transition function of m_j^* such that

$$\tau'_j\big(q_j^*, C\big) = \tau'_j\big(q_j^*, D\big) = q_j^{t_j+1}$$

where τ'_j is the transition function of m'_j. If as a result of this change there is a state q_j, which cannot be reached from the initial state of m_j^* for any sequence of actions by i, then this state is deleted. By our definition of simplicity, m'_j is simpler than m_j^*.

Consider player i. When playing against m_j^* his best response m_i^* generates the following sequence of action pairs starting at period $t_j - l_j$:

$$\mathbf{a}(t_j - l_j, \infty) = \big[a(t_j - l_j, t_j - 1), a(t_j, \infty)\big],$$

where

$$\mathbf{a}(t_j - l_j, t_j - 1) = \big(a_i^*, a_j^*\big), \Big(a_i^{t_j-l_j+1}, a_j^{t_j-l_j+1}\Big), \ldots, \Big(a_i^{t_j-1}, a_j^{t_j-1}\Big) \quad \text{and}$$
$$\mathbf{a}(t_j, \infty) = \big(-a_i^*, a_j^*\big), \Big(a_i^{t_j+1}, a_j^{t_j+1}\Big), \ldots.$$

By Lemma 1, player i is indifferent between playing a_i^* or $-a_i^*$ whenever player j is at state q_j^*. This means that player i is indifferent between a sequence consisting of infinite repetitions of $\mathbf{a}(t_j - l_j, t_j - 1)$ and the sequence $\mathbf{a}(t_j, \infty)$.

Let m'_i denote the machine having $t_j + l_j$ states all of which are connected by constant transitions such that

(1) the machine moves from its $t_j + l_j$ state back to its t_j state, and
(2) the machine carries out the first $t_j - 1$ actions of m_i^* along the path generated by (m_1^*, m_2^*) followed by infinite repetitions of $a_i(t_j - l_j, t_j - 1)$.

It follows that m'_i is a best reply to both m_j^* and m'_j, a contradiction. □

The structure of Nash equilibrium in repeated games with finite machines has been characterized by AR. They show that when each player wants to choose a best response with the minimal number of states against his opponent, there must be a one-to-one correspondence between *states* on the equilibrium path. This means that for every player i and every state q_i on the equilibrium path, there is a unique state q_j such that whenever m_i is in q_i, m_j is in q_j. It turns out that this result also holds when the equilibrium machine of each player is the simplest machine that rationalizes the machine of the opponent.

Proposition 4 (one-to-one correspondence between states on the equilibrium path). *If* (m_1^*, m_2^*) *is an ESF of MG*(δ)*, then there exists a period* t^* *and an integer* $l < t^*$ *such that for* $i = 1, 2$, $q_i^t(m_1^*, m_2^*) = q_i^{t-l}(m_1^*, m_2^*)$ *for* $t \geqslant t^*$.

Proof. See Appendix A. □

It is important to understand that the proof of Proposition 4 is somewhat more involved than the proof of one-to-one correspondence between states in AR. One source of difficulty is the fact that in our model it is not necessary that all states be used on the equilibrium path, and the players' machines may differ in their number of states. Another source of difficulty is related to difference between the proofs of the following claims:

Claim 1. *Suppose* (m_1, m_2) *is a Nash equilibrium of MG*(δ) *with the property that there is no one-to-one correspondence between states on the equilibrium path. Then a machine with fewer states than* m_i *is a best response to* m_j.

Claim 2. *Suppose* (m_1, m_2) *is a Nash equilibrium of MG*(δ) *with the property that there is no one-to-one correspondence between states on the equilibrium path. Then* m_i *is a best response to a machine with fewer states than* m_j.

The proof of the second claim, unlike that of the first, requires us to show that by deleting states from m_j, we have not changed the incentives of player i. That is, we need to make sure that we do not delete any necessary threats or punishments from player j's original machine.

The class of 2×2 games that we are considering can either have two Nash equilibria, (C, D) and (D, C) or (C, C) and (D, D), or a unique Nash equilibrium in which the players coordinate ((C, C) or (D, D)).[2] Suppose we depict the payoff matrix of G as follows:

	C	D
C		
D		.

We call the two sets of action pairs, $\{(C, C), (D, D)\}$ and $\{(C, D), (D, C)\}$, the *main diagonal* and the *off-diagonal* of G, respectively. We can thus say that the Nash equilibria of G is either one of the diagonals, or it is a singleton element in the main diagonal. This classification of the types of Nash equilibria that G may have will be useful in our next results, which characterize the play paths of *ESF*. These results demonstrate that accounting for the complexity of forecasts places severe restrictions on the play paths that can be generated in equilibrium.

Proposition 5. *Suppose the set of Nash equilibria of G coincides with one of the diagonals of this game. If* (m_1, m_2) *is an ESF of MG*(δ) *for some discount factor* δ*, then in every*

[2] This follows from our symmetry assumption, our assumption that D is the minmax action of each player, and the assumption that for $i = 1, 2$, $u_i(C, C) \neq u_i(D_i, C_j)$ and $u_i(C_i, D_j) \neq u_i(D, D)$.

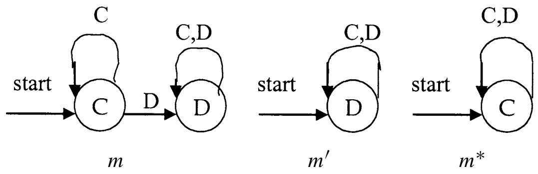

Fig. 7.

period on the equilibrium path players play a Nash equilibrium of G. Conversely, for all $\delta \in [0, 1]$, any play path, in which players play a Nash equilibrium of G every period and which can be generated by a pair of finite machines, is the outcome of some ESF of MG(δ).

The proof of Proposition 5 is given in Appendix A. To understand the intuition for the proof consider the game of Chicken. Let (m_1, m_2) be an *ESF* of the corresponding machine game. Suppose there exists a period t on the equilibrium path in which the outcome is not a Nash equilibrium of G, say (C, C). To prove that this assertion is false, we construct a machine pair (m_1', m_2') such that m_2' is simpler than m_2, and m_1' is a best response to both m_2 and m_2'.

From the assumption that both players choose C at t it follows that each machine must have a non-constant transition in the state that it is at in period t. Therefore, we can simplify m_2, so that its output at q_2^t will be D and it would move to the next state independently of the action chosen by player 1. Denote this simpler machine by m_2'. Let m_1' be a machine that is identical to m_1 except that it has a constant transition at q_1^t. It follows that m_1' is a best response to both m_1 and m_1'. For example, consider the machine m depicted in Fig. 7. For a sufficiently high discount factor, (m, m) is a Nash equilibrium in which (C, C) is played every period. To see why (m, m) is not an *ESF*, note that m' simpler than m, and m^* is a best response to both m and m'.

Proposition 5 has striking implications for the *ESF* of the infinitely repeated Chicken. If for some discount factor a pair of machines is an *ESF* of the infinitely repeated game, then this pair of machines can generate only combinations of the pure Nash equilibrium of the one-shot game.

Proposition 6. *Suppose (D, D) is the unique Nash equilibrium of G. If the discount factor δ sufficiently close to one, then there are only two possible ESF play paths in MG(δ): the play path in which players choose (D, D) each period and the one in which they choose (C, C) each period.*

Proposition 6 implies that in the *ESF* of the infinitely repeated Prisoner's Dilemma the two players must coordinate their actions, i.e., they either always defect or always cooperate. This result is in stark contrast to the Nash folk theorem, which states that every individually rational play path can be sustained as a Nash equilibrium.

The proof of Proposition 6 is given Appendix A. Example 2 in Section 3.2 demonstrates the intuition for why there cannot be an *ESF* play path that consists only of outcomes on the off-diagonal. Here, we provide a simple example that illustrates the intuition for the result that whenever players choose (C, C) they continue to do so in every subsequent period.

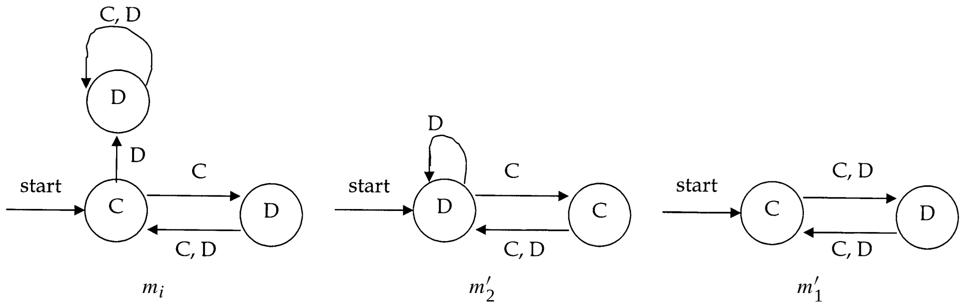

Fig. 8.

Let G be the Prisoner's Dilemma and assume for simplicity that $u(D, D)$ is smaller than the average of $u(C, D)$ and $u(D, C)$. Consider the machine m_i depicted in Fig. 8. For δ sufficiently close to one, (m_1, m_2) is a Nash equilibrium of $MG(\delta)$. However, this machine pair is not an *ESF* of $MG(\delta)$. To see why, note that m_2' is simpler than m_2, and m_1' is a best response to both m_2 and m_2'.

We now turn to characterize the equilibrium paths for the case in which the unique Nash equilibrium of G is (C, C). In stark contrast to the standard folk theorem, when the discount factor is sufficiently close to one, the set of *ESF* play paths is reduced to a subset of the following four sequences:

(S1) (C, C) each period;
(S2) (D, D) in the first period, followed by (C, C) in every subsequent period;
(S3) (D, D) in every odd period and (C, C) in every even period;
(S4) (D, C) each period;
(S5) (C, D) each period.

Proposition 7. *Assume (C, C) is the unique Nash equilibrium of G. Let (m_1, m_2) be an ESF of $MG(\delta)$, where δ is a discount factor sufficiently close to one.*

(i) *Suppose $u_i(C_i, D_j) \geqslant u_i(D_i, C_j)$. Then $\mathbf{a}(m_1, m_2)$ is one of the sequences* (S1)–(S3). *Conversely, each of the sequences* (S1)–(S3) *can be generated by a pair of machines that constitutes an ESF of $MG(\delta)$ for δ sufficiently close to one.*
(ii) *Suppose $u_i(C_i, D_j) < u_i(D_i, C_j)$. Then $\mathbf{a}(m_1, m_2)$ can be any sequence* (S1)–(S5). *Conversely, each of the sequences* (S1)–(S5) *can be generated by a pair of machines that constitutes an ESF of $MG(\delta)$ for δ sufficiently close to one.*

Proposition 7 is proven in Appendix A. The intuition for the proof is similar to the intuition underlying Propositions 5 and 6.

Given Propositions 5–7, it is interesting to compare our results with those of AR. Although our approaches are different, both AR and we obtain a one-to-one correspondence between states *and* actions on the equilibrium paths. However, the set of *ESF* play paths is different from the set obtained by AR. For example, consider the repeated Prisoner's Dilemma. While a play path in which players cooperate every period can be sustained in *ESF*, it cannot be sustained in the equilibrium concept of AR. Consider next the repeated Chicken. While there is no *ESF* in which starting from some point in time, players cooperate every period, such a play path can be obtained in the equilibrium of AR.

5. Equilibrium payoffs

Propositions 5–7 enable us to provide a characterization of the equilibrium payoffs. For this characterization we need to introduce a few notations. Π^{δ}_{ESF} will denote the set of *ESF* payoffs for a discount factor of δ. Let Π^{δ}_{G} denote the set of all payoff pairs obtained from some discounted sum of the Nash equilibrium payoffs of G when the discount factor is δ. That is,

$$\Pi^{\delta}_{G} = \left\{ (\pi_1, \pi_2)\colon \pi_i = (1-\delta)\sum_{t=0}^{\infty} \delta^t u_i\big(\mathbf{a}^t\big) \text{ and } \mathbf{a}^t \in NE(G) \text{ for every } t \right\}.$$

The next pieces of notation represent the payoff pairs associated with the sequences (S1)–(S5) described in Proposition 7. Let $\pi^S(\delta) = (\pi^S_1(\delta), \pi^S_2(\delta))$ where $\pi^S_i(\delta)$ denotes the δ-discounted sum of payoffs that player i obtains from a play path $S \in \{S1, \ldots S5\}$. Thus,

$$\pi^{S1}_i(\delta) = u_i(C, C), \qquad \pi^{S2}_i(\delta) = (1-\delta)u(D, D) + \delta u(C, C),$$
$$\pi^{S3}_i(\delta) = \frac{u(D, D) + \delta u(C, C)}{1+\delta}, \qquad \pi^{S4}_i(\delta) = u_i(D, C), \qquad \pi^{S5}_i(\delta) = u_i(C, D).$$

Finally, to denote the *ESF* payoffs when $NE(G) = \{(D, D)\}$ we use the notations $\pi^{DD}_i(\delta) = u_i(D, D)$ and $\pi^{CC}_i(\delta) = \pi^{S1}_i(\delta)$.

Proposition 8. *Let (m_1, m_2) be an ESF of $MG(\delta)$ for δ sufficiently close to one.*

(i) *If the Nash equilibria of G lie on one of its diagonals, then $\Pi^{\delta}_{ESF} = \Pi^{\delta}_{G}$.*
(ii) *If $NE(G) = \{(C, C)\}$ and $u_i(C_i, D_j) \geqslant u_i(D_i, C_j)$, then $\Pi^{\delta}_{ESF} = \{\pi^S(\delta)\}_{S=S1,\ldots,S3}$.*
(iii) *If $NE(G) = \{(C, C)\}$ and $u_i(C_i, D_j) < u_i(D_i, C_j)$, then $\Pi^{\delta}_{ESF} = \{\pi^S(\delta)\}_{S=S1,\ldots,S5}$.*
(iv) *If $NE(G) = \{(D, D)\}$, then $\Pi^{\delta}_{ESF} = \{\pi^{DD}(\delta), \pi^{CC}(\delta)\}$.*

Proof. Follows directly from Propositions 5–7. □

6. Concluding remarks

This paper should be viewed as a critique of the commonly accepted interpretation of Nash equilibrium as a steady state in which players best respond to correct forecasts about their opponents. The criticism which is raised in this paper is the following: When the players' preferences are affected by their beliefs about others, then the requirement for correct beliefs places severe restrictions on the set of equilibrium outcomes. Thus, when we enrich the standard framework by allowing expectations to enter into the players' preferences, then we can either look for alternative interpretations of Nash equilibria, or look for alternative solution concepts.

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Appendix A

Proof of Proposition 4. Let (m_1, m_2) be an *ESF* of $MG(\delta)$ for some discount factor δ. We begin by introducing a few notations. Let t_i denote the first period on the equilibrium path in which one of the states of m_i appears for the second time. That is, let t_i be the minimal time for which there is a player i and a period $t_i^* < t_i$ such that $q_i^{t_i} = q_i^{t_i^*}$. We say that m_i begins its cyclic phase at t_i^*. We refer to the difference $t_i - t_i^*$ as the "length of the cycle" and denote it by l_i.

Assume there is no one-to-one correspondence between states along the path that is generated by (m_1, m_2). Given Proposition 3, this assumption has two possible implications:

(1) Either the two machines enter their cyclic phases at different periods (i.e., $t_1^* \neq t_2^*$), or
(2) the two machines enter their cyclic phases at the same time, but the length of their cycles differ ($t_1^* = t_2^*$ but $l_1 \neq l_2$).

We consider each of these two cases separately.

Case 1. m_i enters its cyclic phase before m_j: $t_i^* < t_j^*$.

Let t' denote the minimal time in which

(1) both m_i and m_j are in the cyclic phase of the equilibrium path, and
(2) m_i is in the first state of its cyclic phase, (state $q^{t_i^*}$).

Case 1 corresponds to the case in which $t' > t_i^*$.

Construct m_j as follows: Let $S \subset Q_j$ be a subset of states in $Q_j \backslash Q_j(t_i^*, t_j^* - 1)$ such that each $q \in S$ satisfies $\tau_j(q, a) \in Q_j(t_i^*, t_j^* - 1)$ for some $a \in A$. If S is empty, then omit the states in $Q_j(t_i^*, t_j^* - 1)$ from m_j'. Also omit any state that cannot be reached from any of the remaining states. However, if S is nonempty, then for each $q \in S$ and $a \in A$ satisfying $\tau_j(q, a) \in Q_j(t_i^*, t_j^* - 1)$ let $\tau_j'(q, a) = q_j^{t_i^*}$ where $f_j'(q_j^{t_i^*}) = D$ and $\tau_j'(q_j^{t_i^*}, C) = \tau_j'(q_j^{t_i^*}, D) = q_j^{t_i^*}$. Omit all the states in $Q_j(t_i^* + 1, t_j^* - 1)$ and any other state that cannot be reached from any of the remaining states.

If $t_i^* = 1$, let the initial state of m_j' be $q_j^{t_i^*}$. Otherwise, change the transition function of m_j at $q_j^{t_i^* - 1}$ so that the modified machine moves directly into state $q_j^{t'}$ when player i carries out his equilibrium action at $t_i^* - 1$. That is, let the transition function of m_j' satisfy:

$$\tau_j'\left(q_j^{t_i^*-1}, f_i\left(q_i^{t_i^*-1}\right)\right) = q_j^{t'}.$$

If i deviates from his equilibrium action at t_i^*-1, then $\tau'_j(q_j^{t_i^*-1}, -f_i(q_i^{t_i^*-1}))$ depends on whether or not S is empty. If it is, then the transition out of $q_j^{t_i^*-1}$ when player i does not choose $f_i(q_i^{t_i^*-1})$ remains the same as in m_j:

$$\tau'_j\left(q_j^{t_i^*-1}, -f_i\left(q_i^{t_i^*-1}\right)\right) = \tau_j\left(q_j^{t_i^*-1}, -f_i\left(q_i^{t_i^*-1}\right)\right).$$

However, if S is nonempty, then let $\tau_j(q_j^{t_i^*-1}, -f_i(q_i^{t_i^*-1})) = q_j^{t_i^*}$.

From the above changes it follows that $x(m'_j) < x(m_j)$ while $y(m'_j) \leqslant y(m_j)$. Therefore, m'_j is simpler than m_j. By Lemma 1, if m_i is a best response to m_j, then it must also be a best response to m'_j. Thus, (m_1, m_2) cannot be an *ESF* of $MG(\delta)$, a contradiction. □

Case 2. Both m_1 and m_2 enter their cyclic phase at the same time, but have different cycle lengths: $t_i^* = t_j^* = t^*$ but $l_i < l_j$.

By Lemma 1, player i must be indifferent between the equilibrium path from t_i^* onwards and infinite repetitions of the sequence of outcomes from t^* to $t^* + l_i - 1$. We use this observation to construct a simpler machine than m_j to which m_i is a best response.

Let $\widehat{m}_j$ denote the machine obtained when we make the following changes in m_j. Let $\widehat{S} \subset Q_j$ be a subset of states in $Q_j \backslash Q_j(t^* + l_i, t^* + l_j - 1)$ such that each $q \in \widehat{S}$ satisfies $\tau_j(q, a) \in Q_j(t^* + l_i, t^* + l_j - 1)$ for some $a \in A$. If $\widehat{S}$ is empty, then omit the states in $Q_j(t^* + l_i, t^* + l_j - 1)$ from $\widehat{m}_j$. Also omit any state that cannot be reached from any of the remaining states. However, if $\widehat{S}$ is nonempty, then for each $q \in \widehat{S}$ and $a \in A$ satisfying $\tau_j(q, a) \in Q_j(t^* + l_i, t^* + l_j - 1)$ let $\hat{\tau}_j(q, a) = q_j^{t^*+l_i}$ where $\hat{f}_j(q_j^{t^*+l_i}) = D$ and $\hat{\tau}_j(q_j^{t^*+l_i}, C) = \hat{\tau}_j(q_j^{t^*+l_i}, D) = q_j^{t^*+l_i}$. Omit all the states in $(t^* + l_i + 1, t^* + l_j - 1)$ and any other state that cannot be reached from any of the remaining states.

The next set of changes involves the transition function at $q_j^{t^*+l_i-1}$. We change the transition function of m_j so that the modified machine moves directly into the state $q_j^{t^*}$ when player i carries out his equilibrium action at $t^* + l_i - 1$. If i deviates from his equilibrium action at $t^* + l_i - 1$, then $\hat{\tau}_j(q_j^{t^*+l_i-1}, -f_i(q_j^{t^*+l_i-1}))$ depends on whether or not $\widehat{S}$ is empty. If it is, then

$$\hat{\tau}_j\left(q_j^{t^*+l_i-1}, -f_i\left(q_j^{t^*+l_i-1}\right)\right) = \tau_j\left(q_j^{t^*+l_i-1}, -f_i\left(q_j^{t^*+l_i-1}\right)\right).$$

Otherwise, we let

$$\hat{\tau}_j\left(q_j^{t^*+l_i-1}, -f_i\left(q_j^{t^*+l_i-1}\right)\right) = q_j^{t^*+l_i}.$$

From the above changes it follows that $x(m'_j) < x(m_j)$ while $y(m'_j) \leqslant y(m_j)$. Therefore, m'_j is simpler than m_j. Note that by construction, the sequences $\mathbf{a}(m_i, m_j)$ and $\mathbf{a}(m_i, m'_j)$ differ only after period $t^* + l_i - 1$. Also note that the sequence that is generated by (m_i, m'_j) from $t^* + l_i$ onwards consists of infinite repetitions of the outcomes between t^* and $t^* + l_i - 1$ on $\mathbf{a}(m_i, m_j)$. By Lemma 1, if m_i is a best response to m_j, then it is also a best response to m'_j. Thus, (m_1, m_2) cannot be an *ESF* of $MG(\delta)$, a contradiction. □

Each of the Propositions 5–7, is proven in two steps. In the first step we show that given the Nash equilibria of G, any *ESF* play path must satisfy certain properties. Given any play path with those properties, we can construct a pair of *ESF* machines that generate that play path. This is done in the second step of the proof. For this second step we require the notion of a spanning sequence, which we define below.

Let $(\mathbf{a}^t)$ be a sequence of G outcomes, which is generated by some pair of finite machines. It follows that there exists a finite sequence of action pairs $b(K, L)$,

$$b(K, L) = \left(b^1, \ldots, b^K, b^{K+1}, \ldots, b^{K+L}\right)$$

such that a single repetition of $(b^1, \ldots, b^K)$, followed by infinite repetitions of $(b^{K+1}, \ldots, b^{K+L})$, yield the sequence $(\mathbf{a}^t)$. We say that $b(K, L)$ *spans* the sequence $(\mathbf{a}^t)$. We let $b(\underline{K}, \underline{L})$ denote the shortest sequence, which spans $(\mathbf{a}^t)$. Since we are considering only finite machines, there exists a shortest sequence that spans a play path.

Proof of Proposition 5. Suppose the set of Nash equilibria of G is one of the diagonals of this game; that is, the set of Nash equilibria of G is either $\{(C,C),(D,D)\}$ or $\{(C,D),(D,C)\}$. This implies that if (a_1,a_2) is not a Nash equilibrium of G, then no player is best responding to his opponent.

Step 1. Let (m_1,m_2) be an *ESF* of $MG(\delta)$ for some discount factor δ. For each player i denote the output function and the transition function of m_i by f_i and τ_i, respectively. Assume that (m_1,m_2) generates a play path $(\mathbf{a}^t)$ in which there is at least one period in which the outcome is *not* a Nash equilibrium of G. Let Q_1^{ESF} and Q_2^{ESF} be the set of states of m_1 and m_2, respectively that appear on $(\mathbf{a}^t)$. For each player j and for each state $q_j \in Q_j^{ESF}$ let $S_i(q_j) \subseteq Q_i^{ESF}$ be the set of states that satisfy the following: Along the path $(\mathbf{a}^t)$, whenever m_j is in state q_j the machine m_i enters a state in $S_i(q_j)$. By Proposition 4, $S_i(q_j)$ is a singleton. For each $q_j \in Q_j^{ESF}$ we denote the single member of $S_i(q_j)$ by $s_i(q_j)$.

For each player j define Q_j^* to be the set of equilibrium states in Q_j^{ESF} with the property that the pair of actions, $f_j(q_j)$ and $f_i(s_i(q_j))$, is *not* a Nash equilibrium of G (that is, whenever a state in Q_j^* appears on the equilibrium path, the two players do *not* play a Nash equilibrium of G). Thus, for every $q_j \in Q_j^*$,

$$\tau_j(q_j,C) \neq \tau_j(q_j,D).$$

We now show that m_j can be turned into a machine m'_j with the following properties:

(1) each period it chooses an action, which is different from the equilibrium action of m_j at that period, and
(2) it moves from one state to another independently of player i's actions.

Denote the set of states, the output function and the transition function of m'_j by Q'_j, f'_j and τ'_j, respectively. Let $Q'_j = Q_j^{ESF}$. For each $q_j \in Q_j^*$ let $f'_j(q_j) \neq f_j(q_j)$ and

$$\tau'_j(q_j,C) = \tau'_j(q_j,D) = \tau_j\big(q_j, a_i(q_j)\big)$$

where $a_i(q_j) = f_i(s_i(q_j))$. Clearly, $y(m'_j) < y(m_j)$ and $x(m'_j) \leqslant x(m_j)$. This implies that m'_j is simpler than m_j.

Let m'_i be a machine that carries out the equilibrium actions of m_i irrespective of the actions by player j's machine:

(1) $Q'_i = Q_i^{ESF}$.
(2) $f'_i(q_i) = f_i(q_i)$ for each $q_i \in Q'_i$.
(3) For all $q_i \in Q'_i$, $\tau'_i(q_i,C) = \tau'_i(q_i,D) = \tau_i(q_i,a_j)$ where $a_j = f_j(s_j(q_i))$.

Clearly, m'_i is a best response to both m_j and m'_j. Thus, (m_1,m_2) cannot be an *ESF* of $MG(\delta)$, a contradiction.

Step 2. Consider a play path in which every period players play a Nash equilibrium of G. Let $b(\underline{K},\underline{L})$ be the shortest sequence that spans that play path. Consider a machine m_i with $\underline{K}+\underline{L}$ states. For each state q^k, where $k=1,\dots,\underline{K}+\underline{L}$, let the output be b^k. The machine starts at q^1 and passes to the other states according to a transition function with the following properties:

$$\tau_i\big(q^k,C\big) = \tau_i\big(q^k,D\big) = \begin{cases} q^{k+1} & \text{if } k < \underline{K}+\underline{L}, \\ q^{\underline{K}+1} & \text{if } k = \underline{K}+\underline{L}. \end{cases}$$

It is straightforward to verify that (m_1,m_2) is an *ESF* of $MG(\delta)$ for any discount factor δ.

Proof of Proposition 6.

It is easy to see that a pair of single state machines with output D is an *ESF* of $MG(\delta)$ for any discount factor $\delta \in [0,1]$. It is also easy to see (recall Example 1) that a pair of grim trigger machines, which generate a play path with constant cooperation, is an *ESF* for a discount factor sufficiently close to one. It remains to show that when δ is sufficiently close to one, there is no other *ESF* play path.

Let (m_1,m_2) be an *ESF* of $MG(\delta)$, where δ is a discount factor sufficiently close to one. The transition function and the output function of each equilibrium machine is denoted τ_i and f_i, respectively. The following series of claims relate to the path generated by (m_1,m_2).

Claim 6.1. *If both players choose C in the first period, then they must continue to do so in every subsequent period.*

Proof. Assume $\mathbf{a}^1 = (C, C)$ and let t^* be the first period with the following property: Both players choose C in this period ($\mathbf{a}^{t^*} = (C, C)$), but at least one of the players, say j, switches to D in the subsequent period ($\mathbf{a}^{t^*+1} \in \{(C_i, D_j), (D, D)\}$). Assume there exists such a period t^*.

From our assumption that both players cooperate in periods 1 to t^* it follows that m_j has a non-constant transition in each of the states that are visited in those periods: $\tau_j(q_j^t, C) \neq \tau_j(q_j^t, D)$ for $t = 1, \ldots, t^*$ where $\tau_j(q_j^{t^*}, C) = q_j^{t^*+1}$. Moreover, for player i to cooperate at $t = 1$ it must be the case that $\tau_j(q_j^1, D) \neq q_j^1$. From our assumption that player j does not cooperate at $t^* + 1$ it follows that m_j has at least one state distinct from $q_j^{t^*}$. From our conclusion that

$$\tau_j\left(q_j^{t^*}, D\right) \notin \left\{q_j^{t^*}, q_j^{t^*+1}\right\}$$

it follows that m_j has one other state distinct from both $q_j^{t^*}$ and $q_j^{t^*+1}$. Hence, m_j must have at least three states.

Let m'_j be a machine with only two states q_j^D and q_j^C and a transition function τ'_j. The machine starts from state q_j^D where the output is D and the transition function satisfies $\tau'_j(q_j^D, D) = q_j^D$ and $\tau'_j(q_j^D, C) = q_j^C$. In state q_j^C the output is C and once m'_j enters this state, it remains there regardless of the output of m_i, i.e., $\tau'_j(q_j^D, D) = \tau'_j(q_j^D, C) = q_j^C$. From our conclusions regarding the properties of m_j it follows that m'_j is simpler than m_j.

Let m'_i be a machine with the following properties. The set of states of m'_i is $Q'_i = Q_i \cup \{q_i^D\}$ such that both machines start from the same initial state q_i^0. In each of the states $q_i \in Q_i$ the output of both machines is the same, i.e., $f'_i(q_i) = f_i(q_i)$ for all $q_i \in Q_i$. In addition, $\tau'_i(q_i, a_j) = \tau_i(q_i, a_j)$ for all $q_i \in Q_i \backslash \{q_i^0\}$ and $a_j \in \{C, D\}$. At the initial state, which is common to both machines, we have $\tau'_i(q_i^0, C) = \tau_i(q_i^0, C)$ and $\tau'_i(q_i^0, D) = q_i^D$ where q_i^D is a state in which the output is D and the transition function is $\tau'_i(q_i^D, D) = \tau'_i(q_i^D, C) = q_i^D$. Clearly, m'_i is a best response to m'_j for δ sufficiently close to one. Moreover, by construction, m'_i is also a best response to m_j. This contradicts our assumption that (m_1, m_2) is an *ESF* of *MG*(δ). □

Claim 6.2. *If both players choose D in the first period, then they must continue to do so in every subsequent period.*

Proof. Assume $\mathbf{a}^1 = (D, D)$ and let t^* be the first period with the following property: Both players choose D in this period ($\mathbf{a}^{t^*} = (D, D)$), but at least one of the players, say i, switches to C in the subsequent period ($\mathbf{a}^{t^*+1} \in \{(C_i, D_j), (C, C)\}$). Assume there exists such a period t^*.

Let m'_i be a single state machine with output C. Clearly, m'_i is simpler than m_i. Let m'_j be a machine, which is different from m_j only in the following respect:

$$\tau'_j\left(q_j^0, C\right) = q_j^0.$$

It is easy to see that m'_j is a best response to both m_i and m'_i. □

Claim 6.3. *Suppose there exists some period t in which $a^t = (C_i, D_j)$. Let q_j^t be the state of m_j in period t. Then $\tau_j(q_j^t, D) = q_j^t$.*

Proof. Assume that $\tau_j(q_j^t, D) \neq q_j^t$. Then m_j can be simplified by letting it remain in q_j^t, unless player i chooses C. Denote this simpler machine by m'_j. Since δ is sufficiently close to one, it follows that m_i is a best response to both m_j and m'_j, a contradiction. □

Claim 6.4. *If there exists some period t in which $\mathbf{a}^t = (C_i, D_j)$, then $t > 1$.*

Proof. Suppose $\mathbf{a}^1 = (C_i, D_j)$. We show that (m_1, m_2) cannot be an *ESF* of *MG*(δ). Since D is a dominant strategy the transition from the initial state of m_j must satisfy:

$$\tau_j\left(q_j^0, D\right) \neq \tau_j\left(q_j^0, C\right).$$

By Claim 6.3, $\tau_j(q_j^0, D)$ must be equal to q_j^0. Thus, m_i can be simplified such that m_j will be a best response to both m_i and its simplification. To see why, let m_i' be a single state machine with a constant output of D. Clearly, m_i' is simpler than m_i, and m_j is a best response to m_i'. Since this contradicts our assumption that (m_1, m_2) is an *ESF* of *MG*(δ), it must be the case that $\mathbf{a}^1 \neq (C_i, D_j)$. □

Claim 6.5. *There exists no period in which the outcome is* (C_i, D_j).

Proof. Let t^* be the earliest period in which one player chooses C while the other chooses D. By Claims 6.1 and 6.2, the outcome in the first period cannot be (C, C) or (D, D). Hence, $t^* = 1$, in contradiction to Claim 6.4. □

From Claims 6.1–6.5 it follows that the only possible *ESF* play paths are ones in which the players coordinate each period. □

Proof of Proposition 7.

Step 1. Let (m_1, m_2) be an *ESF* of *MG*(δ) where δ is a discount factor that is close to one. Let t_i^* denote the first period on the equilibrium path in which machine i enters a state that had already appeared before t_i^*. By Proposition 4, $t_1^* = t_2^* = t^*$.

Claim 7.1. *If there exists a period* $t < t^*$ *in which both players choose* C*, then both players continue to choose* C *in every period between* t *and* t^*.

Proof. Assume the claim is false, so that on the equilibrium path there is a period $t \leqslant t' < t^*$ in which the outcome is (C, C), while at $t' + 1$ at least one player, say j, chooses D. Let $q_j^{t'}$ be the state of m_j at t'. Consider making the following change in m_j. First, change the output of m_j in state $q_j^{t'}$ from C to D, and let the machine remain in that state for every action by player i. Second, omit the states in $Q_j(t' + 1, t^*)$ from m_j'. Also omit any state that cannot be reached from any of the remaining states. Let $S \subset Q_j(1, \ldots, t' - 1)$ such that for each $q \in S$ there exists $a(q) \in A$ satisfying $\tau_j(q, a(q)) \in Q_j(t' + 1, t^*)$. If S is nonempty, then for each $q \in S$ let $\tau_j'(q, a(q)) = q_j^{t'}$.

Denote the machine that results from this change by m_j'. Clearly, $x(m_j') < x(m_j)$ and $y(m_j') < y(m_j)$, and so m_j' is simpler than m_j.

Consider next a machine m_i', which is identical to m_i except for the following difference: The machine remains in state $q_i^{t'}$ unless the output of player j's machine is C, in which case m_j' moves to state $q_i^{t'+1}$ (i.e., $\tau_i'(q_i^{t'}, D) = q_i^{t'}$, while $\tau_i'(q_i^{t'}, C) = q_i^{t'+1}$). It is easy to see that m_i' is a best response to both m_j' and to m_j, in contradiction to the assumption that (m_1, m_2) is an *ESF* of *MG*(δ). □

Claim 7.2. *Suppose that on the equilibrium path there is a period* t *in which the outcome is* (D_i, C_j)*. Then the outcome in every subsequent period must also be* (D_i, C_j).

Proof. Let $\hat{t}$ be the smallest t that satisfies: There exists i such that

$$\mathbf{a}^t(m_1, m_2) = (D_i, C_j), \qquad \mathbf{a}^{t+1}(m_1, m_2) \neq (D_i, C_j).$$

Assume the claim is false so that a period $\hat{t}$, as defined above, exists on the play path of (m_1, m_2). Let $q_i^{\hat{t}}$ be the state of m_i at $\hat{t}$. From Proposition 4 it follows that regardless of the output of j's machine, m_i does not remain in $q_i^{\hat{t}}$ in period $\hat{t} + 1$.

By Claim 7.1, there is no period before $\hat{t}$ in which the outcome is (C, C). Thus, in every period t before $\hat{t}$ the outcome is either (D, D) or (D_i, C_j). It follows that the transition function of m_i must satisfy the following:

$$\tau_i\big(q^t, C\big) = \begin{cases} q^t & \text{if } \mathbf{a}^t(m_i, m_j) = (D, D), \\ \tau_i\big(q^t, D\big) & \text{if } \mathbf{a}^t(m_i, m_j) = (D_i, C_j). \end{cases}$$

This has three important implications. First, the only state in m_i, that appears on the equilibrium path before $\hat{t}$ and which has a transition into $q_i^{\hat{t}}$, is $q_i^{\hat{t}-1}$. Second, none of the states, which appear on the equilibrium path before $\hat{t}$, have a transition into a state that does not appear on the path in the first $\hat{t}$ periods: For every $t < \hat{t}$ and $a \in \{C, D\}$, $\tau_i(q^t, a) \in \{q^t, q^{t+1}\}$.

Suppose we make the following changes in m_i. First, we change its output at $q_i^{\hat{t}}$ from D to C. Second, we change the transition function at $q_i^{\hat{t}}$ such that the machine remains in that state for every action by player j. Finally, we delete any state that does not appear in the first $\hat{t}$ periods of the equilibrium path. Denote the resulting machine by m_i'. Then $x(m_i') < x(m_i)$ and $y(m_i') < y(m_i)$, which means that m_i' is simpler than m_i.

Consider a machine m_j', which is identical to m_j except perhaps for the transition function at $q_j^{\hat{t}}$: m_j' remains in that state unless player i chooses D, in which case, m_j' moves to $q_j^{\hat{t}+1}$.

To complete the proof we need to show that the assumption that a period $\hat{t}$ exists necessarily leads to a contradiction. We show this by proving that m_j' is a best response to both m_i and m_i'. We first show that m_j' is a best response to m_i. Note that the only difference between m_j and m_j' can be the transition out of state $q_j^{\hat{t}}$ when the output of player i's machine is C. Since the output of m_i at $\hat{t}$ is D, both (m_i, m_j) and (m_i, m_j') generate the same play path. Since by assumption, m_j is a best response to m_i, so is m_j'.

We now show that m_j' is a best response to m_i'. First, note that m_i' includes all the states of m_i that appear on the play path of (m_i, m_j) before $\hat{t}$. Moreover, both the transition function and the output function of m_i' at those states are the same as in m_i. Second, note that the first $\hat{t} - 1$ outcomes in the play path of (m_i, m_j) are also the first $\hat{t} - 1$ outcomes in the play path of (m_i', m_j'). However, from $\hat{t}$ onwards player j prefers the play path of (m_i', m_j') to that of (m_i, m_j). In addition, when m_i' reaches $q_i^{\hat{t}}$ it is optimal for player j to constantly play C. Hence, from the result that m_j' is a best response to m_i, and from the fact that the output of m_j' at $q_j^{\hat{t}}$ is C and m_j' remains in that state when player i chooses C, it follows that m_j' is a best response to m_i'. □

Claim 7.3. *Let t be the earliest period on the equilibrium path that satisfies $t < t^*$ and $\mathbf{a}^t = (C, C)$. Then $t \leqslant 2$.*

Proof. Assume $t > 2$. By Claims 7.1 and 7.2, the first two outcome on the equilibrium path must be (D, D). Since C is dominating for each player, both m_i and m_j must have non-constant transitions in the first two states on the equilibrium path. Therefore, $y(m_j) \geqslant 2$. It is also easy to see that $x(m_j) \geqslant 2$. We now construct a pair of machines (m_i', m_j') as follows.

- m_j' has three states, q^0, q^D and q^C. It begins at q^0 where the output is C. It moves to q^D if the initial move of i is C. Otherwise, it moves to state q^C. The output at q^D is D, and if m_j' reaches that state, it remains there for every action of player i. Similarly, the output at q^C is C, and if m_j' reaches that state, it remains there for every action of player i. The machine m_j' is depicted in Fig. 9. It follows that $x(m_j') = 2$ but $y(m_j') = 1$. Therefore, m_j' is simpler than m_j.
- m_i' is constructed by adding a state to m_i and changing the transition function at m_i's initial state. Let q_i^0 and q_i^1 be the two states of m_i such that q^0 is the initial state, and q^1 is the state that follows q^0 on the equilibrium path. If the initial action of player j's machine is D, m_i' moves to state q_i^1. However, if player j's initial move is C, then m_i' moves to a new state (one that is not in m_i) in which the output is C. m_i' remains in that new state for every action of player j. Figure 10 displays an example of a three-state m_i and the corresponding m_i'.

By construction, m_i' is a best response to both m_j and m_j'. This implies that (m_i, m_j) cannot be an *ESF* of *MG*(δ), a contradiction. □

Claim 7.4. *If $u_i(C_i, D_j) \geqslant u_i(D_i, C_j)$, there exists no period on the equilibrium path in which the outcome is (D_i, C_j).*

Proof. Assume there exists some period on $\mathbf{a}(m_1, m_2)$ in which the outcome is (D_i, C_j). Let t^* be the earliest period with that property. By Claim 7.2, the outcome in every subsequent period must also be (D_i, C_j). This

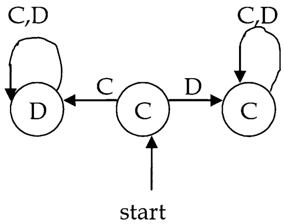
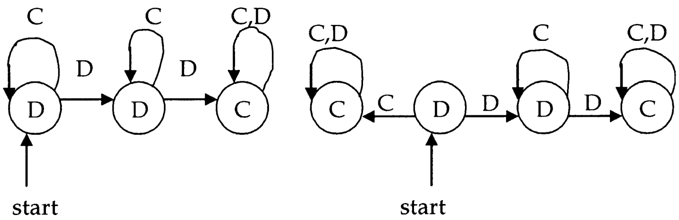


Fig. 9. The machine m'_j.



m_i m'_i

Fig. 10.

implies that player i necessarily prefers to have the outcome (D_i, C_j) from t^* onwards to having the outcome (C_i, D_j) onwards. This in turn implies that $u_i(C_i, D_j) < u_i(D_i, C_j)$, a contradiction. □

Suppose there exist a pair of periods such that the outcome in one is (C, C), while the outcome in the other is (D_i, C_i). Let t and t' be the earliest periods with this property, so that $\max\{t, t'\} < t^*$. Let $(a_i^t, a_j^t) = (C, C)$ and $(a_i^{t'}, a_j^{t'}) = (D_i, C_j)$. Assume $t < t'$. By Claim 7.1, $(a_i^s, a_j^s) = (C, C)$ for all $t^* > s > t$, a contradiction. Assume $t > t'$. By Claim 7.2, $(a_i^s, a_j^s) = (D_i, C_j)$ for all $s > t$, a contradiction. Thus, $\mathbf{a}(m_1, m_2)$ can be of only two types:

(1) Each period the players miscoordinate, i.e., the outcome each period is either (D, C) or (C, D).
(2) Each period the players coordinate, i.e., the outcome each period is either (C, C) or (D, D).

Suppose $\mathbf{a}(m_1, m_2)$ is of the first type. Then by Claim 7.3, $\mathbf{a}(m_1, m_2)$ consists of infinite repetitions of either (C, D) or (D, C). Suppose $\mathbf{a}(m_1, m_2)$ is of the second type. Then by Claims 7.1 and 7.3, $\mathbf{a}(m_1, m_2)$ can be one of three types:

(1) (C, C) each period.
(2) (D, D) in the first period, followed by infinite repetitions of (C, C).
(3) Infinite repetitions of the sequence $(D, D), (C, C)$.

Step 2. Let $((a_1^t, a_2^t))_{t=1}^{\infty}$ be a sequence of action-pairs that can be generated by a pair of finite machines.

Suppose $(a_1^t, a_2^t) = (D_i, C_j)$ for all t. Let (m_i, m_j) be the following pair of machines. m_i has a single state in which the output is D. m_j has two states, the initial state q^C and a "punishment state" q^D. The output at q^C is C, and m_j remains in this state unless player i chooses C, in which case m_j moves to q^D. The output at q^D is D, and m_j remain in this state regardless of what player i's actions. Clearly, $\mathbf{a}(m_i, m_j) = ((a_1^t, a_2^t))_{t=1}^{\infty}$. If $u_i(C_i, D_j) < u_i(D_i, C_j)$, then by m_i is a best response to m_j for δ sufficiently close to one. Furthermore, it is easy to verify that m_j is the simplest machine that induces player i to constantly choose D. It follows that (m_i, m_j) is an *ESF* of *MG*(δ) for $\delta \geqslant \delta^*$.

Suppose next that $((a_1^t, a_2^t))_{t=1}^{\infty}$ consists only of outcomes on the main diagonal (i.e., (C, C) or (D, D)). There are three possible play paths. For each possible path, we construct a pair of identical machines (m^*, m^*) that generate that path.

Case A. Suppose players choose (C, C) each period. Let m^* be a single state machine with an output of C. Clearly, (m^*, m^*) is an *ESF* of *MG*(δ) for any discount factor.

Case B. Suppose the initial outcome is (D, D), followed by a constant play of (C, C). Let m^* have two states, q^D and q^C. The output at q^D is D, while the output at q^C is C. The machine begins at q^D and remains there unless the opponent's initial move is D. In that case m^* moves to state q^C where it stays regardless of the actions of the other player.

If δ is sufficiently close to one, then m^* is a best response against itself. Let m' denote a machine that is simpler than m^*. If $x(m') < x(m^*)$ (i.e., m' has only one state), then clearly any best response to m' would require a player to choose C each period. If $x(m') = x(m^*)$, then $y(m') = 0$. This means any best response to m' cannot have an output function that assigns the action D to the initial state. It follows that (m^*, m^*) must be an *ESF* of *MG*(δ) for δ sufficiently close to one.

Case C. Suppose the path consists of infinite repetitions of the sequence $(D, D), (C, C)$. The machine here is almost identical to the one described in point 2 above. The only difference is, that from state q^C the machine returns to state q^D regardless of the action chosen by the other player. By using the same arguments as in Case B, one can verify that $(m^*.m^*)$ is an *ESF* of *MG*(δ) for δ sufficiently close to one.

This completes the proof of Proposition 7. □

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