Supplementary Appendix to Collective Information Acquisition

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September 2020

1 The variance cost function

In this appendix we show that our analysis extends to the case in which the cost of a signal $\{(q_j, r_j)\}_{j=1}^{J}$ is proportional to the *variance* of the posteriors on the state $\omega = 1$, where the mean posterior is the prior (i.e., $\sum_{j=1}^{J} q_j \cdot r_j = p$.). I.e.,

$$c\left(\{(q_j, r_j)\}_{j=1}^{J}\right) = \kappa \cdot \sum_{j=1}^{J} q_j \left(r_j - p\right)^2 .$$

Note that Lemma 1 extends to this cost function. To see why, define $h(r) \equiv (r-p)^2$ and note that this function is convex in r. Then, the same arguments in the proof of Lemma 1 readily apply to the newly defined function $h(r)$.

It follows that we can restrict attention to signals that are represented by the triplet (q, r_H, r_L) as in the main text. We therefore consider the cost function

$$c(q, r_H, r_L) = \kappa \cdot \left(q \cdot (r_H - p)^2 + (1-q)(r_L - p)^2\right).$$

Substituting $r_L = \frac{p - q r_H}{1-q}$, we can rewrite the cost as a function of only q and r_H :

$$c(q, r_H) = \kappa \frac{q}{1-q} (r_H - p)^2 .$$

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To simplify the exposition, we focus on the case where $\kappa > n$. This guarantees that information is never "too cheap" so that removing all the uncertainty (i.e., $r_H = 1$ and $r_L = 0$) becomes optimal (the KL-divergence cost function in our main text satisfies this for all (n, κ)). The analysis remains essentially the same when $\kappa \leq n$, but the exposition is more cumbersome since we have to take care of more corner solutions in the designer's optimization problem.

To establish that the qualitative analysis in the main text extends to the above cost function, we mimic the steps in the proof of Proposition 2. First, we find three functions, $r_H^*(\theta, \lambda), r_L^*(\theta, \lambda)$ and $q^*(\theta, \lambda)$, that satisfy non-wastefulness and maximize the Lagrangian that is given by Equation (18), for any multiplier λ and any profile of types θ. Second, we show that for *any* $\lambda \geq 0$, the function $q^*(\theta, \lambda)$ is increasing in each player's type while $r_H^*(\theta, \lambda)$ and $r_L^*(\theta, \lambda)$ are decreasing in each player's type. Hence, the function $Q^*(\theta_i, \lambda)$ that is induced by $q^*(\theta, \lambda)$ (according to Equation 16) is monotone. Third, we apply an argument in Hellwig (2013) that guarantees the existence of some $\lambda^* \geq 0$ for which the mechanism defined by $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ has a non-negative virtual surplus. It then follows that the functions $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ define the mechanism that attains the maximal aggregate surplus (Equation 14) subject to (i) $Q(\theta_i)$ is monotone and (ii) the aggregate virtual surplus (Equation 15) is non negative.

PART I. The first-order condition with respect to q for an interior solution $(\tilde{q}, \tilde{r}_H)$ that maximizes the Lagrangian $\mathcal{L}(q, r_H, \lambda)$ is

$$r_H - (1 - w) - \frac{1}{n} \cdot \kappa \cdot \left(\frac{r_H - p}{1 - q}\right)^2 = 0 \tag{FOCq}$$

while the first-order condition with respect to r_H is

$$q - \frac{1}{n} \cdot 2\left(\kappa \frac{q}{1 - q}(r_H - p)\right) = 0 \tag{FOCr}$$

where w is as defined in the main text. From the second equation we have that $\frac{r_H - p}{1-q} = \frac{n}{2\kappa}$. Plugging this into the first equation yields:

$$r_H - (1 - w) - \frac{1}{n} \cdot \kappa \cdot \left(\frac{n}{2\kappa}\right)^2 = 0$$

Hence,

$$\tilde{r}_H(w) = \frac{n}{4\kappa} + 1 - w \tag{1}$$

Using $\frac{r_H - p}{1-q} = \frac{n}{2\kappa}$ again we can solve for $\tilde{q}$:

$$\tilde{q}(w) = \frac{1}{2} + \frac{2\kappa}{n}(w - (1-p)) \tag{2}$$

Finally, from $r_L = \frac{p - q r_H}{1-q}$ it follows that

$$\tilde{r}_L(w) = 1 - w - \frac{n}{4\kappa} \tag{3}$$

Since $\mathcal{L}$ is not a concave function in general, it is not a priori guaranteed that $(\tilde{q}, \tilde{r}_H)$ is a maximizer of $\mathcal{L}$. Our first result establishes that although $\mathcal{L}$ is not concave, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful then it is indeed a maximizer of $\mathcal{L}$.

Lemma A0. For any value of r_H and w, define $\hat{q}(r_H, w)$ to be the value of q that satisfies $\mathcal{L}_1(q, r_H; w) = 0$ (i.e. FOCq) and define $\hat{r}_L(r_H, w) \equiv \frac{p - r_H \cdot \hat{q}(r_H, w)}{1 - \hat{q}(r_H, w)}$ to be value of r_L that is determined by r_H and $\hat{q}(r_H, w)$. Then, $\hat{\mathcal{L}}(r_H, w) \equiv \mathcal{L}(\hat{q}(r_H, w), r_H\ ;\ w) = \kappa(p - r_L)^2$.
Proof. By definition we have that

$$\hat{\mathcal{L}}(r_H, w) = \hat{q}(r_H, w) \cdot c_1(\hat{q}(r_H, w), r_H) - c(\hat{q}(r_H, w), r_H)$$

Substituting

$$\begin{aligned}
\hat{q}(r_H, w) &= \frac{p - \hat{r}_L(r_H, w)}{r_H - \hat{r}_L(r_H, w)} \\
c_1(\hat{q}(r_H, w), r_H) &= \kappa \cdot \left(\frac{r_H - p}{1 - \hat{q}(r_H, w)} \right)^2 \\
&= \kappa \cdot \left(\frac{r_H - p + \hat{q}(r_H, w) r_H - \hat{q}(r_H, w) r_H}{1 - \hat{q}(r_H, w)} \right)^2 \\
&= \kappa \cdot (r_H - \hat{r}_L(r_H, w))^2 \\
c(r_H, \hat{r}_L(r_H, w)) &= \kappa \cdot \left(\hat{q}(r_H, w) \cdot (r_H - p)^2 + (1 - \hat{q}(r_H, w))(\hat{r}_L(r_H, w) - p)^2 \right) \\
&= \kappa \cdot [\frac{p - \hat{r}_L(r_H, w)}{r_H - \hat{r}_L(r_H, w)} \cdot (r_H - p)^2 + \frac{r_H - p}{r_H - \hat{r}_L(r_H, w)} \cdot (p - \hat{r}_L(r_H, w))^2] \\
&= \kappa(r_H - p)(p - \hat{r}_L(r_H, w))
\end{aligned}$$

We obtain that

$$\begin{aligned}
\hat{\mathcal{L}}(r_H, w) &= \frac{p - \hat{r}_L(r_H, w)}{r_H - \hat{r}_L(r_H, w)} \cdot \kappa \cdot (r_H - \hat{r}_L(r_H, w))^2 - \kappa(r_H - p)(p - \hat{r}_L(r_H, w)) \\
&= \kappa(p - \hat{r}_L(r_H, w))^2.
\end{aligned}$$

□

Lemma A1. *For any w, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful, then it maximizes* $\mathcal{L}(q, r_H\ ;\ w)$.

Proof. First, we show that $(\tilde{q}, \tilde{r}_H)$ is a local maximum of $\mathcal{L}(q, r_H; w)$. Then, we show that $\mathcal{L}(\tilde{q}, \tilde{r}_H; w)$ is greater than the value of $\mathcal{L}$ in any corner solution.

To show that $(\tilde{q}, \tilde{r}_H)$ is a local maximum it suffices to show that (i) $\mathcal{L}_{11}(q, r_H; w) < 0$ and (ii) the determinant of the Hessian of $\mathcal{L}(q, r_H; w)$ is positive, when evaluated at $(\tilde{q}, \tilde{r}_H)$. To establish (i), note that

$$\mathcal{L}_{11}(q, r_H; w) = \frac{d}{dq}\left(r_H - (1-w) - \frac{1}{n}\cdot\kappa\cdot\left(\frac{r_H - p}{1-q}\right)^2\right) = -\frac{2}{n}\kappa\frac{(p - r_H)^2}{(1-q)^3} < 0$$

To establish (ii) note that

$$\mathcal{L}_{22}(q, r_H; w) = \frac{d}{dr_H}\left(q - \frac{1}{n}\cdot 2\left(\kappa\frac{q}{1-q}(r_H - p)\right)\right) = -\frac{2}{n}q\frac{\kappa}{1-q}$$
$$\mathcal{L}_{12}(q, r_H; w) = \frac{d}{dq}\left(q - \frac{1}{n}\cdot 2\left(\kappa\frac{q}{1-q}(r_H - p)\right)\right) = 1 - \frac{2\kappa(r_H - p)}{n(q-1)^2}$$

The determinant of the Hessian is equal to $\mathcal{L}_{11}(q, r_H; w)\cdot\mathcal{L}_{22}(q, r_H; w) - (\mathcal{L}_{12}(q, r_H; w))^2$, which reduces to

$$\left(\frac{2\kappa}{n}\cdot\frac{r_H - p}{1-q}\right)^2\cdot\frac{q}{(1-q)^2} - \left(1 - \frac{1}{1-q}\cdot\frac{2\kappa}{n}\cdot\frac{r_H - p}{1-q}\right)^2$$

At $(\tilde{q}, \tilde{r}_H)$ we have $\frac{\tilde{r}_H - p}{1-\tilde{q}} = \frac{n}{2\kappa}$, and hence, the determinant reduces to

$$\frac{\tilde{q}}{(1-\tilde{q})^2} - \frac{\tilde{q}^2}{(1-\tilde{q})^2} > 0.$$

We now turn to show that $\mathcal{L}(\tilde{q}, \tilde{r}_H; w)$ is (weakly) greater than the value of $\mathcal{L}$ in any corner solution. To see this, recall that q can take any value from 0 to $\frac{p}{r_H}$ and that

$$\mathcal{L}(q, r_H; w) = q[r_H - (1-w)] - \frac{1}{n}\cdot c(q, r_H)$$

If $q = 0$, then no signal is acquired and hence, $\mathcal{L}(0, r_H; w) = 0$. On the other hand, $\mathcal{L}(\tilde{q}, \tilde{r}_H; w)$ can be written as

$$\left[\frac{p - \tilde{r}_L}{\tilde{r}_H - \tilde{r}_L}\cdot\frac{\kappa}{n}\cdot(\tilde{r}_H - \tilde{r}_L)^2 - \frac{1}{n}\kappa(\tilde{r}_H - p)(p - \tilde{r}_L)\right] = \kappa(p - \tilde{r}_L)^2 \geq 0 \tag{4}$$

Suppose next that $\mathcal{L}(\tilde{q},\tilde{r}_H;w) < \max_{r_H\in(p,1]} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right)$. Since by (4), $\mathcal{L}(\tilde{q},\tilde{r}_H;w) \geq 0$ while

$$\mathcal{L}\left(\frac{p}{r_H}, r_H; w\right) = \frac{p}{r_H}[r_H - (1-w)] - \frac{\kappa}{n} \cdot \frac{p}{r_H - p}(r_H - p)^2 \tag{5}$$

it must be that any $r'_H \in \arg\max_{r_H} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right)$ is greater or equal to $1-w$ (otherwise, $\max_{r_H\in(p,1]} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right) < 0$,a contradiction). The expression on the R.H.S. of (5) has a unique maximizer equal to $\sqrt{\frac{n}{k}(1-w)}$. For this to be greater or equal to $1-w$ it must be that $n \geq \kappa$, a contradiction.

The only remaining corner solution is $r_H = 1$. Recall that by Lemma A0, $\hat{\mathcal{L}}(r_H, w) = \kappa(p - \hat{r}_L(r_H, w))^2$. The fact that $\hat{\mathcal{L}}$ is maximized at $\tilde{r}_H$ implies that $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H(w) < 1$. It follows that

$$\hat{\mathcal{L}}(1, w) = \kappa(p - \hat{r}_L(1, w))^2 < \kappa(p - \hat{r}_L(\tilde{r}_H, w))^2 = \hat{\mathcal{L}}(\tilde{r}_H, w).$$

Since $\hat{\mathcal{L}}(\tilde{r}_H, w)$ and $\hat{\mathcal{L}}(1, w)$ are the values of the Lagrangian when r_H attains the values $\tilde{r}_H$ and 1, respectively (where q is optimally determined according to FOCq) the proof is complete. □

Lemma A2. *If an interior solution $(\tilde{q}, \tilde{r}_H)$ exists but is wasteful, then in the optimal solution, $r^*_H = 1 - \theta^{(n-m+1)}$.*

Proof. For any value of r_H and w, define $\hat{q}(r_H, w)$ to be the value of q that satisfies $\mathcal{L}_1(q, r_H; w) = 0$ and define $\hat{r}_L(r_H, w) \equiv \frac{p - r_H \cdot \hat{q}(r_H, w)}{1 - \hat{q}(r_H, w)}$ to be value of r_L that is determined by r_H and $\hat{q}(r_H, w)$. Therefore,

$$r_H - (1-w) - \frac{\kappa}{n} \cdot \left(\frac{r_H - p}{1 - \hat{q}(r_H, w)}\right)^2 = 0$$

Solving for $\hat{q}(r_H, w)$ yields

$$\hat{q}(r_H, w) = 1 - \frac{r_H - p}{\sqrt{\frac{n}{\kappa} \cdot (r_H + w - 1)}}$$

Since $\hat{r}_L(r_H, w) = (p - \hat{q}(r_H, w) \cdot r_H)/(1 - \hat{q}(r_H, w))$ the above equation is equivalent to

$$r_H - (1-w) - \frac{\kappa}{n}(r_H - \hat{r}_L(r_H, w))^2 = 0$$

We can therefore solve for $\hat{r}_L$ to obtain

$$\hat{r}_L(r_H, w) = r_H - \sqrt{\frac{n}{\kappa}\left[r_H - (1-w)\right]} \tag{6}$$

From (6) we can derive the following three properties of the function $\hat{r}_L(r_H, w)$:

(P1) *For any w, the function $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H$.* Since

$$\frac{\partial}{\partial r_H}\hat{r}_L(r_H, w) = 1 - \frac{1}{2}\sqrt{\frac{n}{\kappa}}\left(r_H + w - 1\right)^{-\frac{1}{2}}$$

we have that

$$\frac{\partial}{\partial r_H}\hat{r}_L(r_H, w) = 0 \Longleftrightarrow 1 = \frac{n}{4\kappa}\cdot\frac{1}{r_H + w - 1} \Longleftrightarrow r_H = \frac{n}{4\kappa} + 1 - w = \tilde{r}_H(w)$$

Since

$$\frac{\partial^2}{\partial r_H \partial r_H}\hat{r}_L(r_H, w) = \frac{1}{4}\left(r_H + w - 1\right)^{-\frac{3}{2}} \geq 0$$

we have that $\tilde{r}_H(w)$ is a minimum point.

(P2) *For any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H.* This follows from $\frac{\partial^2}{\partial r_H \partial r_H}\hat{r}_L(r_H, w) \geq 0$.

(P3) *For any r_H, the function $\hat{r}_L(r_H, w)$ is decreasing in w.* This follows from the R.H.S. of (6).

We have thus established that for any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H and attains minimum at $\tilde{r}_H$. Hence, for all values of $r_H \geq 1 - \theta^{(n-m+1)} > \tilde{r}_H$ the function $\hat{r}_L(r_H, w)$ is increasing in r_H. Recall that $\hat{\mathcal{L}}(r_H, w) = \kappa(p - \hat{r}_L(r_H, w))^2$ where $\hat{\mathcal{L}}(r_H, w)$ is the value of the Lagrangian for any r_H, when q is determined according to (FOCq). It follows that when r_H is restricted to the domain $\left[1 - \theta^{(n-m+1)}, 1\right]$ the maximum of $\hat{\mathcal{L}}$ is attained when $r_H = 1 - \theta^{(n-m+1)}$. Thus, $r_H^*(\theta_1, \theta_2) = 1 - \theta^{(n-l+1)}$, which completes the proof. $\square$

PART II. We now turn to show that $q^*(\theta, \lambda)$ is increasing in each player's type while $r_H^*(\theta, \lambda)$ and $r_L^*(\theta, \lambda)$ are decreasing in each player's type. Fix θ_{-i} and λ. Suppose that $\theta_i' > \theta_i$ and denote $w \equiv w(\theta_i, \theta_{-i}, \lambda)$ and $w' \equiv w(\theta_i', \theta_{-i}, \lambda)$ so that $w' > w$.

If $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ is not interior, then no signal is acquired when the agents report θ, i.e. $q^*(\theta, \lambda) = 0$ and $r_L^*(\theta, \lambda) = p$. Without loss of generality we can assume that in this case $r_H^*(\theta, \lambda) = 1$, and it immediately follows that $q^*(\theta', \lambda) \geq q^*(\theta, \lambda)$ and $r_L^*(\theta', \lambda) \leq r_L^*(\theta, \lambda)$ and $r_H^*(\theta', \lambda) \leq r_H^*(\theta, \lambda)$.

We therefore assume that $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ is interior. As we explain in the main text, this also implies that $(\tilde{r}_H(\theta', \lambda), \tilde{q}(\theta', \lambda))$ is interior. Note that: (1) $\tilde{r}_H$ and $\tilde{r}_L$, as given by Equations (1) and (3) are decreasing in $w(\theta_i, \theta_{-i}, \lambda)$, (2) $w(\theta_i, \theta_{-i}, \lambda)$ is increasing in θ_i and (3) $\tilde{q}$ is decreasing in $\tilde{r}_L$ and decreasing in $\tilde{r}_H$ for all $\tilde{r}_L \leq p \leq \tilde{r}_H$. These properties, together with Lemmas A1 and A2, ensure that the remainder of the proof is the the same as in the proof of Proposition 2, with the obvious adjustments to the case of the variance cost.

PART III. From the Lemmas A1 and A2, it follows that for any $\lambda \geq 0$ and for each profile of types θ, the values $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ that maximize $\mathcal{L}(q, r_H; w)$ satisfy that $q^*(\theta, \lambda)$ is unique and $r_H^*(\theta, \lambda)$ is unique whenever $q^*(\theta, \lambda) > 0$ (i.e., whenever a signal is purchased). We have also established that $q^*(\theta, \lambda)$ is monotone in any θ_i. It remains to show there exist $\lambda \geq 0$ for which $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ induce a non-negative expected aggregate virtual surplus. This follows from the same arguments given in the proof of Proposition 2.

This completes our proof.

2 Properties of the cost function

$$
\begin{aligned}
\frac{c\left(q,r_H\right)}{\kappa} &= q\left(r_H\log\frac{r_H}{p}+\left(1-r_H\right)\log\frac{1-r_H}{1-p}\right)\\
&+\left(1-q\right)\left(\frac{p-q\cdot r_H}{1-q}\log\frac{\frac{p-q\cdot r_H}{1-q}}{p}+\left(1-\frac{p-q\cdot r_H}{1-q}\right)\log\frac{1-\frac{p-q\cdot r_H}{1-q}}{1-p}\right)\\
&= q\left(r_H\log\frac{r_H}{p}+\left(1-r_H\right)\log\frac{1-r_H}{1-p}\right)+\left(1-q\right)\left(r_L\log\frac{r_L}{p}+\left(1-r_L\right)\log\frac{1-r_L}{1-p}\right)\\
\frac{c_1\left(q,r_H\right)}{\kappa} &= r_H\left(\ln\frac{r_H}{\frac{p-qr_H}{(1-q)}}\right)+\left(1-r_H\right)\left(\ln\frac{1-r_H}{1-\frac{p-qr_H}{(1-q)}}\right)\\
&= r_H\left(\ln\frac{r_H}{r_L}\right)+\left(1-r_H\right)\left(\ln\frac{1-r_H}{1-r_L}\right)\\
\frac{c_2\left(q,r_H\right)}{\kappa} &= -q\left(\ln\frac{1}{r_H}\left(r_H-1\right)\frac{\frac{p-qr_H}{(1-q)}}{\frac{p-qr_H}{(1-q)}-1}\right)=q\left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right)\\
\frac{c_{11}\left(q,r_H\right)}{\kappa} &= \frac{\left(p-r_H\right)^2}{\left(1-q\right)^3\left(1-\frac{p-qr_H}{(1-q)}\right)\frac{(p-qr_H)}{(1-q)}}=\frac{\left(p-r_H\right)^2}{\left(1-q\right)^3\left(1-r_L\right)r_L}=\frac{1}{r_L}\frac{\left(r_H-r_L\right)^2}{\left(1-r_L\right)\left(1-q\right)}\\
\frac{c_{22}\left(q,r_H\right)}{\kappa} &= \frac{q}{r_H\left(1-r_H\right)}\frac{-\frac{(p-qr_H)}{(1-q)}-\frac{(1-q)qr_H(1-r_H)}{(1-q)^2}+\frac{(p-qr_H)^2}{(1-q)^2}}{\frac{(p-qr_H)}{1-q}\left(\frac{p-qr_H}{1-q}-1\right)}\\
&= \frac{q}{r_H\left(1-r_H\right)}+\frac{q}{r_L\left(1-r_L\right)}\cdot\frac{q}{1-q}\\
\frac{c_{12}\left(q,r_H\right)}{\kappa} &= \left(\ln\frac{r_H\left(\frac{p-qr_H}{(1-q)}-1\right)}{\left(r_H-1\right)\frac{(p-qr_H)}{(1-q)}}\right)+\frac{q\left(p-r_H\right)}{\left(1-q\right)^2\left(\frac{p-qr_H}{(1-q)}\right)\left(\frac{p-qr_H}{(1-q)}-1\right)}\\
&= \left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right)+\frac{q\left(p-r_H\right)}{\left(1-q\right)^2\left(r_L\right)\left(r_L-1\right)}\\
&= \left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right)+\frac{q}{r_L\left(1-r_L\right)}\frac{r_H-r_L}{(1-q)}
\end{aligned}
$$