A Model of Competing Narratives: Online Appendix

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This appendix contains proofs omitted from the main file.

Proof of Proposition 1

Consider an auxiliary two-player game. Player 1's strategy space is D, and α denotes an element in this space. Player 2's strategy space is $\Delta(\mathcal{G} \times D)$, and σ denotes an element in this space. The payoff of player 1 from the strategy profile (α, σ) is $-[\alpha - \sum_{G,d} \sigma(G,d)d]^2$. The payoff of player 2 from (α, σ) is equal to $\sum_{G,d} \sigma(G,d)\widetilde{U}(G,d;\alpha)$, where $\widetilde{U}(G,d;\alpha) = U(G,d;\alpha)$ if $V(G,\alpha;\alpha) = \mu$ and $\widetilde{U}(G,d;\alpha) = -\infty$ otherwise.

Note that $\sum_{G,d} \sigma(G,d)d = \alpha(\sigma)$ by definition. Therefore, when player 1 chooses α to best-reply to σ, we have $\alpha = \alpha(\sigma)$. Non-nullness ensures that $\mathcal{G}$ includes a DAG G^* that induces $V(G,\alpha;\alpha) = \mu$. It follows that when player 2 chooses σ to best-reply to α, it maximizes $U(G,d;\alpha)$ subject to $V(G,\alpha;\alpha) = \mu$. Therefore, a Nash equilibrium in this auxiliary game is equivalent to our notion of equilibrium.

Our objective is thus to establish existence of a Nash equilibrium (α, σ) in this auxiliary game. Since p_G is a continuous function of α, so is U. In addition, the strategy spaces and payoff functions of the two players in the auxiliary game satisfy standard conditions for the existence of Nash equilibrium. ■

Proof of Step 2 in the proof of Proposition 4

Let G be the lever DAG $a \to x \to y$. Denote $p_{ay} \equiv p(x=1 \mid a,y)$. Our objective is to find the maximal values for $p_G(y=1 \mid a=1)$ and $p_G(y=1 \mid a=0)$ subject to the constraint that either $p_{a^*1} = p_{a^*0} \in \{0,1\}$ for some a^*, or $p_{1,y^*} = p_{0,y^*} \in \{0,1\}$ for some y^*. We use the shorthand notation $\alpha = \alpha(\sigma)$.

Recall that

$$p_G(y=1|a=1) = p(x=1|a=1)p(y=1|x=1) + p(x=0|a=1)p(y=1|x=0)$$

and by NSQD,

$$p_G(y=1|a=0) = \frac{\mu - \alpha p_G(y=1|a=1)}{1-\alpha}$$

Since we are free to choose what outcome of x to label as 1 or 0, there are four cases to consider.

Case 1. Let $X_{a=1,x=1}$ be the set of lever variables that satisfy $p_{11} = p_{10} = 1$. It follows that for every $x \in X_{a=1,x=1}$, $p(x=1|a=1) = 1$ while $p(x=0|a=1) = 0$. Hence,

$$\begin{aligned} \max_{x \in X_{a=1,x=1}} p_G(y &= 1|a=1) = \max_{x \in X_{a=1,x=1}} p(y=1|x=1) \\ \max_{x \in X_{a=1,x=1}} p_G(y &= 1|a=0) = \frac{\mu - \alpha \min_{x \in X_{a=1,x=1}} p_G(y=1|x=1)}{1-\alpha} \end{aligned}$$

where

$$p(y=1|x=1) = \frac{\alpha\mu + (1-\alpha)\mu p_{01}}{\alpha\mu + (1-\alpha)\mu p_{01} + \alpha(1-\mu) + (1-\alpha)(1-\mu)p_{00}}$$

The R.H.S. of this equation is maximized when $p_{01} = 1$ and $p_{00} = 0$, and it is minimized when $p_{01} = 0$ and $p_{00} = 1$. Therefore,

$$\max_{x \in X_{a=1,x=1}} p_G(y=1|a=1) = \frac{\mu}{\mu + \alpha(1-\mu)}$$

where this maximum is attained by $p_{11} = p_{10} = p_{01} = 1$ and $p_{00} = 0$ (which is equivalent to a lever variable defined as $x = y + a(1-y)$, while

$$\max_{x \in X_{a=1,x=1}} p_G(y=1|a=0) = \frac{\mu - \alpha \frac{\alpha\mu}{\alpha+(1-\alpha)(1-\mu)}}{1-\alpha} = \frac{\mu(\alpha+1-\mu)}{1-\mu(1-\alpha)}$$

where this maximum is attained by $p_{11} = p_{10} = p_{00} = 1$ and $p_{01} = 0$ (which is equivalent to a lever variable defined as $x = a + (1-a)(1-y)$).

Case 2. Let $X_{a=0,x=0}$ be the set of lever variables that satisfy $p_{01} = p_{00} = 0$. Hence,

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=0) = \max_{x \in X_{a=0,x=0}} p(y=1|x=0)$$

and by NSQD,

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=1) = \frac{\mu - (1-\alpha)\min_{x \in X_{a=0,x=0}} p(y=1|x=0)}{\alpha}$$

where

$$p(y=1|x=0) = \frac{\alpha\mu(1-p_{11}) + (1-\alpha)\mu}{\alpha\mu(1-p_{11}) + (1-\alpha)\mu + \alpha(1-\mu)(1-p_{10}) + (1-\alpha)(1-\mu)}$$

Since the R.H.S. of this equation *decreases* in p_{11} and *increases* in p_{10} we have that

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=0) = \frac{\mu}{\mu + (1-\alpha)(1-\mu)}$$

which is attained by $p_{01} = p_{00} = p_{11} = 0$ and $p_{10} = 1$ (which is equivalent to a lever variable $x = a(1-y)$), while

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=1) = \frac{\mu - (1-\alpha)\frac{(1-\alpha)\mu}{(1-\alpha)\mu+(1-\mu)}}{\alpha} = \frac{\mu(2-\alpha-\mu)}{1-\alpha\mu}$$

which is attained by $p_{01} = p_{00} = p_{10} = 0$ and $p_{11} = 1$ (which is equivalent to a lever variable $x = ay$).

Case 3. Let $X_{y=1,x=1}$ be the set of lever variables that satisfy $p_{01} = p_{11} = 1$. Hence,

$$\max_{x \in X_{y=1,x=1}} p_G(y=1|a=1) = \max_{x \in X_{y=1,x=1}} p(x=1|a=1)p(y=1|x=1)$$

By NSQD,

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=0) = \frac{\mu - \alpha \min_{x \in X_{y=1,x=1}} p(x=1|a=1)p(y=1|x=1)}{1-\alpha}$$

where for $x \in X_{y=1,x=1}$,

$$p(x=1|a=1)p(y=1|x=1) = (\mu+(1-\mu)p_{10}) \cdot \frac{\mu}{\mu + \alpha(1-\mu)p_{10} + (1-\alpha)(1-\mu)p_{00}}$$

Since the R.H.S. of this equation is *increasing* in p_{10} and *decreasing* in p_{00} it follows that

$$\max_{x \in X_{y=1,x=1}} p_G(y=1|a=1) = \frac{\mu}{\mu + \alpha(1-\mu)}$$

which is attained by $p_{01} = p_{11} = p_{10} = 1$ and $p_{00} = 0$ (which is equivalent to a lever variable $x = y + a(1-y)$), whereas,

$$\min_{x \in X_{y=1,x=1}} p_G(y=1|a=1) = \frac{\mu^2}{\mu + (1-\alpha)(1-\mu)}$$

which is attained by $p_{01} = p_{11} = p_{00} = 1$ and $p_{10} = 0$ (which is equivalent to a lever variable $x = y + (1-y)(1-a)$) such that

$$\max_{x \in X_{a=0,x=0}} p_G(y=1|a=0) = \frac{\mu}{\mu + (1-\alpha)(1-\mu)}$$

Case 4. Let $X_{y=0,x=0}$ be the set of lever variables that satisfy $p_{00} = p_{10} = 0$. Maximizing $p_G(y=1|a=1)$ is equivalent to minimizing $1 - p_G(y=0|a=1)$. Since $p(y=0|x=1) = 0$ it follows that

$$p_G(y=0|a=1) = p(x=0|a=1)p(y=0|x=0)$$

where

$$
\begin{aligned}
p(x &= 0|a=1) = \mu(1-p_{11}) + (1-\mu) = 1-\mu p_{11} \\
p(y &= 0|x=0) = \frac{1-\mu}{1-\mu+\alpha\mu(1-p_{11})+(1-\alpha)\mu(1-p_{01})} \\
&= \frac{1-\mu}{1-\mu(\alpha p_{11}+(1-\alpha)p_{01})}
\end{aligned}
$$

Hence, we want to find p_{11} and p_{01} that minimize

$$
\frac{(1-\mu)(1-\mu p_{11})}{1-\mu(\alpha p_{11}+(1-\alpha)p_{01})}
$$

This expression *increases* in p_{01} and *decreases* in p_{11}. Therefore,

$$
\max_{x\in X_{y=0,x=0}} p_G(y=1|a=1) = 1-\frac{(1-\mu)^2}{1-\alpha\mu} = \frac{\mu(2-\alpha-\mu)}{1-\alpha\mu}
$$

which is attained by $p_{10} = p_{00} = p_{01} = 0$ and $p_{11} = 1$ (which in turn is equivalent to a lever variable $x = ay$)

Similarly,

$$
\max_{x\in X_{y=0,x=0}} p_G(y=1|a=0) = 1-\min_{x\in X_{y=0,x=0}} p_G(y=0|a=0)
$$

where

$$
\begin{aligned}
p_G(y &= 0|a=0) = p(x=0|a=0)p(y=0|x=0) \\
&= \frac{(1-\mu)[(1-\mu)+\mu(1-p_{01})]}{(1-\mu)+(1-\alpha)\mu(1-p_{01})+\alpha\mu(1-p_{11})}
\end{aligned}
$$

Since the R.H.S. of this expression *decreases* in p_{01} and *increases* in p_{11}, we have that

$$
\max_{x\in X_{y=0,x=0}} p_G(y=1|a=0) = 1-\frac{(1-\mu)^2}{1-\mu(1-\alpha)} = \frac{\mu(1+\alpha-\mu)}{1-\mu(1-\alpha)}
$$

which is attained by $p_{10} = p_{00} = p_{11} = 0$ and $p_{01} = 1$ (which is equivalent to a lever narrative $x = y(1-a)$).

From the above four cases we obtain two candidate lever variables for maximizing $p_G(y = 1|a = 1)$: $x = ay$ and $x = y + a(1 - y)$. The latter leads to a higher expected anticipatory payoff if and only if

$$\frac{\mu}{\mu + \alpha(1 - \mu)} > \frac{\mu(2 - \alpha - \mu)}{1 - \alpha\mu}$$

which holds if and only if $\mu < 1 - \alpha$. Similarly, we obtain two candidate lever variables for maximizing $p_G(y = 1|a = 0) : x = y(1 - a)$ and $x = y + (1 - y)(1 - a)$. The latter leads to a higher expected anticipatory payoff if and only if

$$\frac{\mu}{\mu + (1 - \alpha)(1 - \mu)} > \frac{\mu(1 + \alpha - \mu)}{1 - \mu(1 - \alpha)}$$

which holds if and only if $\mu < \alpha$.